Chapter 7. Continuous Point Location

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7.1. Location Introduction

Location Theory is concerned with the formulation and solution of location problems, i.e. *where to place or locate an object or facility*. An enormous variety of location problems exist from the very large (locating a new manufacturing complex or military base) to the very small (arranging silverware and china on a table or placing integrated circuits on a printed circuit board). A large body of literature has been dedicated to solving these location problems.

The emphasis in this class is on obtaining efficiently computable solutions to models of practical interest. Interesting theoretical developments without practical applications or situations where mathematical analysis is not useful will not be covered in detail. The lectures will focus on problem formulation, model construction, and solution algorithms of real life location problems.

There is a strong geometrical component to location problems. A picture can usually provide a large amount of insight in the problem and suggest a solution. Similarly, algorithms that exploit the geometrical properties tend to be more efficient.

Examples of Location Problems

- Several new distribution centers for a national company.
- A new fire station for a growing suburban county.
- Reorganization of a manufacturing facility.
- Determination of lane depths in a block stacking warehouse.

- Placement of garment components on a cloth roll.
- Placement of workstations along a microload WIP system.
- Automatic pallet building from boxes arriving on a conveyor.
- Placing sprinklers on the lawn.
- Military aircraft loading.
- Assignment of storage locations to products in a warehouse.
- Solving a traditional cardboard puzzle.
- Selecting a parking space in a mall parking lot.

Classification of Location Problems

Introduction

When faced with a location problem, the following series of questions should be asked:

1. what kind of object has to be located,
2. how is the target location area structured,
3. what are the objective(s) and costs parameters, and
4. what are the constraint(s)?

Based upon the answer to these questions, the location problem can be classified, the model formulated, and a solution procedure selected.

Dimensionality and Structure of the Object to be Located

Object Dimensionality

Volume Location is concerned with locating three-dimensional objects. An example of volume location is the loading of trucks or aircraft or the building of pallet loads out of boxes.

Area Location is concerned with locating two-dimensional objects. An example of area location is determining the department layout of a manufacturing facility or the location of garment components on a roll of raw material.

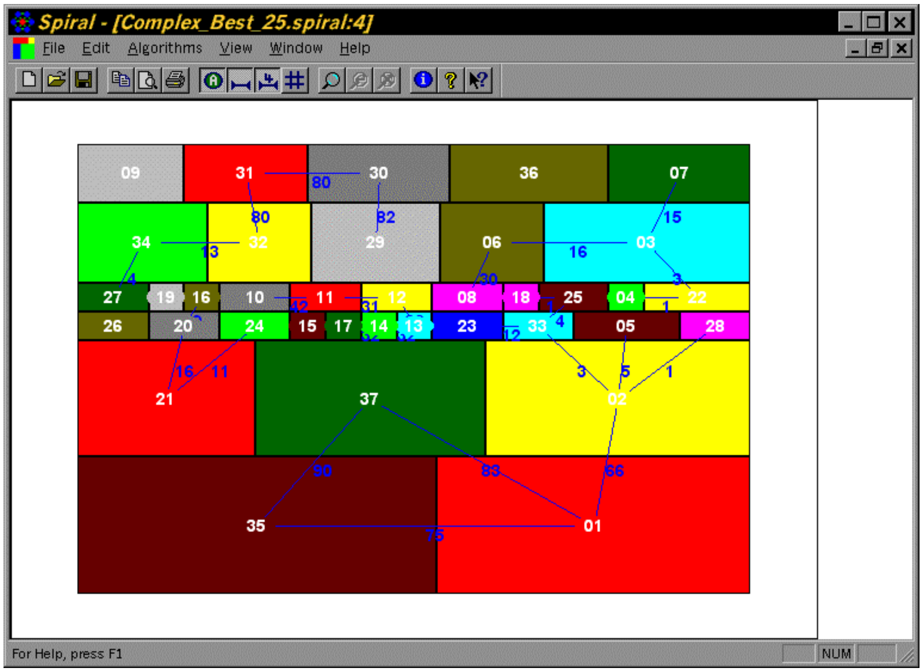


Figure 7.1. Facilities Design as an Example of Two-Dimensional Location

Line Location is concerned with locating one-dimensional objects. An example of line location is determining the picking zones for order picking of both sides of a wide aisle to a cart driving on the centerline of the aisle.

Figure 7.2. Order Picking Zone Design as an Example of One-Dimensional Location

Point Location is concerned with locating zero-dimensional objects. Point location is used any time the size of the object is negligible compared to the size of the target area. This case accounts for the large majority of location problems and location algorithms. Frequent applications are industrial distribution systems such as locating a new distribution center.

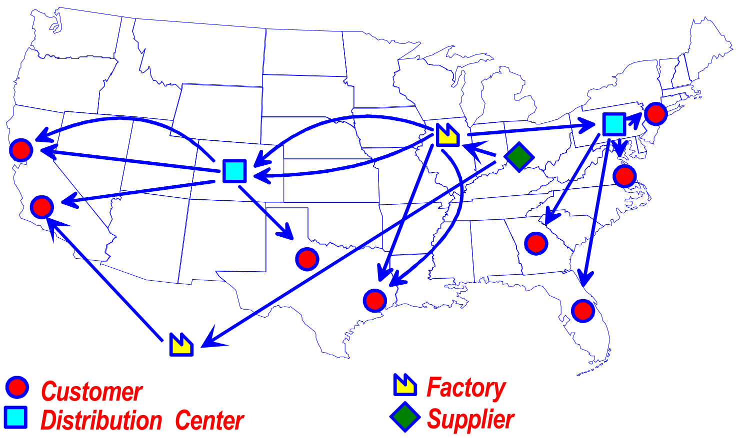


Figure 7.3. Supply Chain Design as an Example of Zero-Dimensional Location

Higher dimensional location problems do exist but are rare. If the constraints or parameters change over time, then a temporal dimension has to be added to the problem and this class of problems is referred to as ***dynamic location***. Other characteristics of the location can be modeled as constraints. For example, the loading of goods on an aircraft requires that the goods fit in three dimensions but also that their weight is balanced along the fuselage and perpendicular to the fuselage.

Object Structure

The structure or lack thereof of the object to be located is also important. For example, a manufacturing facility with a very long and narrow shape is usually not acceptable. Hence, it is important that not only there is enough ground space for a manufacturing department but also that it is of the right shape.

Objects such as liquids and gases have no internal structure, while objects such as boxes on a pallet have a completely defined structure. Manufacturing departments in plant layout usually have a partial structure, since their shape can vary between certain boundaries. In warehouse operations, the size of a shipment can sometimes be reduced by nesting the items efficiently in the shipping container. These nesting characteristics are an example of relevant object structure in location decisions.

Consider the case of facilities design where the shape ratio of the minimum enclosing rectangle of a department must be less than a certain limit to avoid long and narrow departments. If yt_k, yb_k, xr_k, xl_k are the top, bottom, right, and left coordinates of the minimum enclosing rectangle and S_k is the maximum shape ratio, then the following constraints on the length l_k and width w_k must hold for each department k.

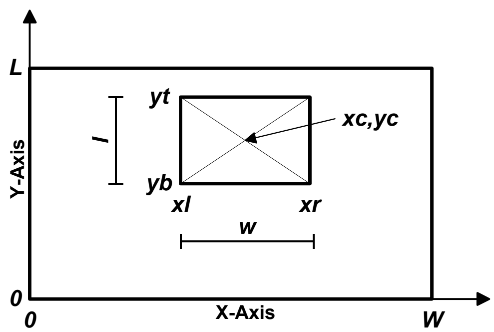


Figure 7.4. Variable and Parameter Illustration

$$
\begin{aligned}
w_k &= xr_k - xl_k \\
l_k &= yt_k - yb_k \\
l_k &\le S_k w_k \\
w_k &\le S_k l_k
\end{aligned}
\qquad (7.1)
$$

Number of Object to be Located (Single versus Multiple)

Location problems can have one (***single facility***) or more (***multifacility***) objects to be located.

Structure of the Target Area for Point Location

Continuous Location (Planar and Spherical)

The target location area is a plane or a sphere without any further structure. The number of possible locations is then infinitely large. The distance costs are based on a distance norm (since a distance table would be also infinitely large). The models are continuous and usually can be analyzed fairly efficiently. Typical applications are the rough-cut location of distribution centers for national companies.

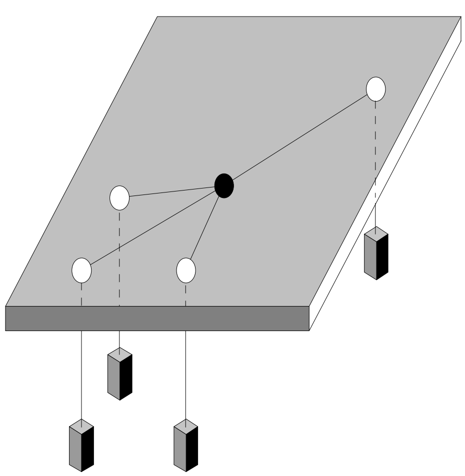


Figure 7.5. Varignon Frame as an Example of Continuous Location

Grid or Subdivision Location

The target location area is a plane, subdivided in a number of (equal square) areas. The number of candidate locations is finite, but very large. Distance costs are based on a distance norm. A typical example is the assignment of unit loads of different products to storage locations in a warehouse. For example, assigning 100,000 products to 200,000 possible locations using discrete locations would create 20,000,000,000 binary assignment variables, which is clearly too large to be a practical formulation and to yield a reasonable solution method.

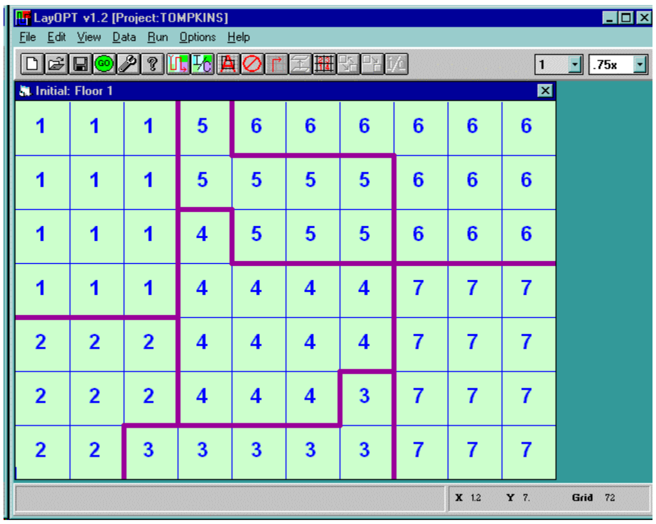


Figure 7.6. Discrete Facilities Design as an Example of Grid Location

Network and Tree Location

The target location area is a network, i.e. a collection of nodes and arcs or edges. If there is no special network structure that can be exploited, then the models are solved by the algorithms used in discrete point location. If the network has no cycles, i.e. it is a tree, then as a general rule specialized and efficient algorithms exist.

The primary application of network location models is when the problem is based upon a transportation network, such as roads, railroads, or waterways.

Figure 7.7. Distribution Center Location as an Example of Network Location

Discrete Candidate Locations

The target location area is a collection of discrete candidate locations. The number of the candidate locations is finite and usually fairly small. These models are the most realistic location models, but the associated computational and data collection costs are very high. Actual distances can be used in the objective and constraints and complex regions with obstacles and infeasible areas can be incorporated.

Typical applications are the detailed design of distribution networks for national companies.

Location Costs

Feasibility versus Optimization

For many location problems the first and overriding goal is to obtain a feasible solution, i.e., a solution that satisfies all the constraints. Once such a feasible solution has been found, the secondary goal is to find a "better" solution, i.e., to optimize with respect to some objective function. The two most common types of objective functions are described next.

Minisum versus Minimax Objective Functions

The ***minisum objective*** consists of the sum of the individual cost components and the objective is thus to optimize the overall or average performance. This objective is appropriate and used in business systems and is also called ***economic efficiency***. This problem is also called the ***median*** problem on networks.

The objective function for the minisum problem can then be written as

$$\min_X \left\{ \sum_j C_j(X) \right\} \tag{7.2}$$

where X indicates the coordinates of the new object to be located, j is the index of the existing and fixed objects and $C_j(X)$ denotes the cost of locating the new object at X with respect to the existing object j.

The ***minimax objective*** consists of the largest individual cost component of an existing facility and the objective is thus to optimize the worst case behavior. This objective is often used in military, emergency, and public sector systems and is also called ***economic equity***. This problem is also called the ***center*** problem on networks.

The objective function for the minimax problem can then be written as

$$\min_X \left\{ \max_j C_j(X) \right\} \tag{7.3}$$

For example, assume 4 points located on a line at positions 0, 5, 6 and 7, respectively. Assume further that the cost of serving each of these points is strictly proportional to the distance between these points and the new facility. The optimal location of the new facility with respect to the minisum objective is the median of these points, i.e. $X^* = 5.5$, so that as many points are to the left as to the right. Actually, the line segment between five and six contains an infinite number of alternative median locations. The optimal location with respect to the minimax objective is the center of these points, i.e. $X^* = 3.5$, so that the distance to the leftmost and rightmost point is equal.

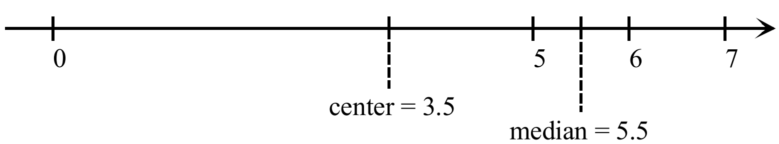


Figure 7.8. Median versus Center Illustration

Observe that the optimal median location would not change if the leftmost point was located at -1000 rather than at 0. For this particular example, where the location domain is a line, the order of the fixed locations is important rather than their actual location. Also observe that the optimal center location would not change if an additional 1000 points were located between coordinates 5 and 6. The center location is always determined by a number of "extreme" locations and the number and location of all the other "interior" objects does not matter.

A third type of objective function often used in the location of "obnoxious" facilities such as waste water treatment plants or military installations is the maximin objective, where objects are located in such way as to maximize the minimum distance.

The objective function of the maximin problem can then be written as:

$$\max_{X}\left\{\min_{j} C_j(X)\right\} \tag{7.4}$$

If we call the optimal solution to the maximim problem then anti-center then the location of the anti-center in the previous example is 2.5 as illustrated in the following figure. The ***Maximin objective*** consists of the smallest individual cost component of an existing facility and the objective is thus to optimize the worst-case behavior. This objective is often used in military, emergency, and public sector systems and is also called ***economic equity***.

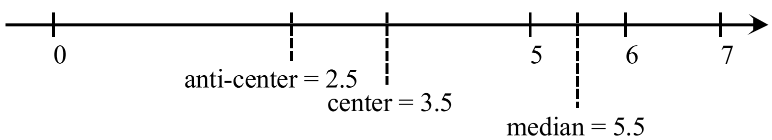


Figure 7.9. Center versus Anti-Center Illustration

Fixed Weights versus Variable Weights

Location problems have fixed weights or relationships if the relationship between the new and existing facilities does not depend on the location of the new facility but is fixed in advance. These problems are also called pure location problems.

If the relationship or weight depends on the location of the new facility, then these weights themselves become variables and these problems are called ***location-allocation*** problems. An example is the assignment of customers to the closest distribution center, moving the distribution center might not only increase the distance to a customer but also assign this customer to another distribution center.

Linear versus Concave Costs (Fixed costs or not)

A location problem is said to have ***linear costs*** is the cost increase linearly with the distance to existing facilities.

A location problem is said to have ***fixed costs*** or ***concave costs*** if the overall location cost consists of a linear function plus a fixed cost. The overall location cost curve is then concave. Economies of scale as well as different technologies or different sizes of the objects to be located generate concave cost curves.

The most common concave objective function is a piecewise linear concave function with an offset at the origin. The offset corresponds to a fixed cost and the breakpoints correspond to cost discounts in function of the distance.

Interaction between Objects to be Located

An important distinction is if the to be located objects have interactions among themselves or only with already existing facilities. If there are multiple new facilities with interactions among themselves then the objective function typically becomes a quadratic or higher order function. The problem of locating two-dimensional departments in a block layout in a facilities design project typically has a quadratic objective function. The number of new facilities is sometimes indicated by p.

Deterministic or Stochastic

If the cost (and parameters) are given by a single value then the problem is said to be deterministic. If on the other hand either costs or parameters are sample from a probability distribution then the problem is said to be stochastic. In the design of distribution systems the customer demand is usually stochastic, but it is approximated by its deterministic mean value.

Static or Dynamic

A location problem is said to be static or single period if costs and parameters do not change over time. A problem is said to be dynamic is the costs and or parameters change over time.

Location Constraints

Capacitated versus Uncapacitated

If the capacity of a new facility to serve old facilities is limited then the location problem is called ***capacitated***, otherwise ***uncapacitated***.

Infeasible Regions

If certain regions of the solution space are not feasible for the location of the objects then the problem is said to contain infeasible regions. An example is the location of distribution centers in the continental US, where the Great Lakes and the Gulf of Mexico are infeasible regions.

7.2. Euclidean Location

Euclidean Distance Norm

Definition

A distance norm is the formula for computing the distance between two points in the plane. Let d_{ij} denote the distance between two points i and j in the plane with coordinates (x_i, y_i) and (a_j, b_j), respectively. The Euclidean norm is then computed as

$$d_{ij}^E = \sqrt{(x_i - a_j)^2 + (y_i - b_j)^2} \tag{7.5}$$

The superscript E denotes the Euclidean distance norm. The ***Euclidean*** distance is also called the straight line travel and is frequently used in national distribution problems and for communications problems where straight line travel is an acceptable approximation. The actual over the road distances in national distribution problems can then be approximated by multiplying the Euclidean distance with an appropriate factor, e.g. 1.2 for continental United States or 1.26 for the South Eastern United States.

Euclidean Norm Properties

Since it is proper distance norm, the Euclidean norm satisfies the four properties of any distance norm:

Non-Negativity

$$d^E(X) \geq 0 \qquad \forall X \tag{7.6}$$

Equality to Zero

$$d^E(X) = 0 \Leftrightarrow X = 0 \tag{7.7}$$

Homogeneity

$$d^E(kX) = |k| d(X) \qquad \forall X \tag{7.8}$$

This property is sometimes also called scalability.

Triangle Inequality

$$d^E(X) + d^E(Y) \geq d^E(X+Y) \qquad \forall X, \forall Y \tag{7.9}$$

Symmetry

$$d^E(-X) = d^E(X) \qquad \forall X \tag{7.10}$$

This property follows from the homogeneity with k= -1.

In addition to the distance properties, the Euclidean norm satisfies the following properties.

Continuous

A function is said to be continuous at point *a* if the following condition holds

$$\forall \delta > 0 \rightarrow \exists \varepsilon > 0 : if \left\| x - a \right\| < \varepsilon \Rightarrow \left\| f(x) - f(a) \right\| \leq \delta \tag{7.11}$$

Convex

A function is said to be convex if the chord or line segment connecting any two points on the graph of the function never lies below the graph of the function, i.e., if the following condition holds.

$$d^E(\lambda X_1 + (1-\lambda) X_2) \leq \lambda d^E(X_1) + (1-\lambda) d^E(X_2) \qquad \lambda \in [0,1] \tag{7.12}$$

If the second derivative of the function is defined everywhere in the domain of the function, then a function is said to be convex if the second derivative is nonnegative everywhere in the domain of the function.

A function is said to be strictly convex if the above inequality holds as a strict inequality for any pair of distinct X_1 and X_2 and any $\lambda \in (0,1)$. In other words, strictly convex means that the line segment lies strictly above the graph of the function except at the two endpoints of the line segment. The Euclidean distance norm is not strictly convex.

Differentiable except at the fixed facilities

The gradient of the Euclidean distance is given by

$$\begin{aligned} \frac{\partial d^E(X)}{\partial x_i} &= \frac{(x_i - a_j)}{\sqrt{(x - a_j)^2 + (y - b_j)^2}} \\ \frac{\partial d^E(X)}{\partial y_i} &= \frac{(y_i - b_j)}{\sqrt{(x - a_j)^2 + (y - b_j)^2}} \end{aligned} \tag{7.13}$$

This gradient is not defined at the location of the existing facilities (a_j, b_j).

Single Facility Minisum Location

Introduction

Varignon Frame

Weber (1909) published his treatment of industrial location and in an appendix the use of the *Varignon Frame* was described as a way to solve the single facility minisum Euclidean location problem. The Varignon Frame is illustrated in Figure 7.10.

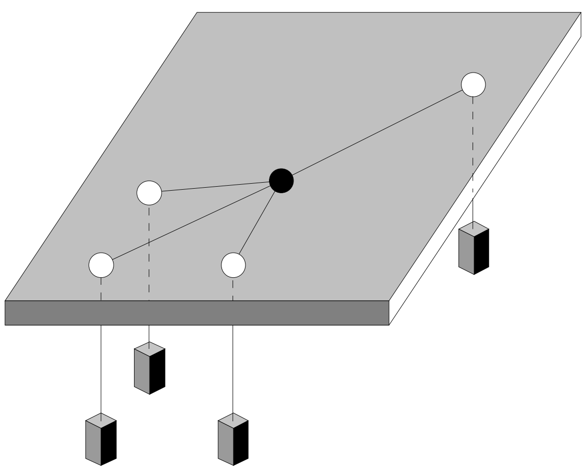


Figure 7.10. Varignon Frame

The optimal location of the knot can be found based on the principles of statics. The position of the knot is such that the (vector) sum of all forces on it equals zero. Projecting the force vectors on the x and y axes gives us two equations. The variables are illustrated in Figure 7.11.

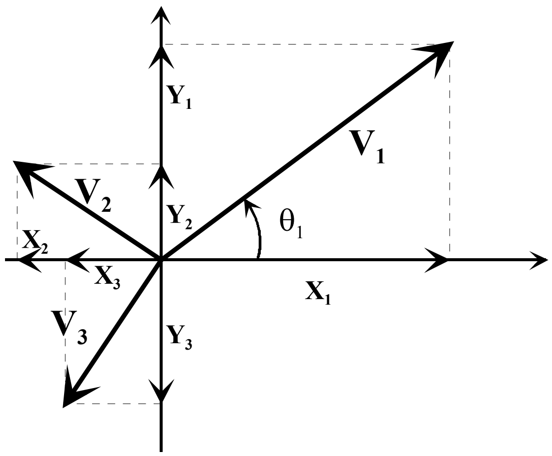


Figure 7.11. Varignon Force Schematic

To determine the equilibrium position, the horizontal and vertical components of the forces need to be balanced. The projection of the force vectors on the horizontal and vertical axis are based on the point coordinates. This is illustrated in the next figure.

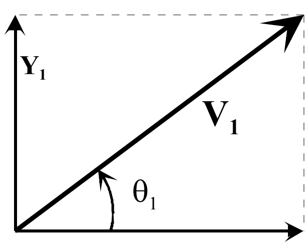


Figure 7.12. Varignon Force Projection

$$\begin{aligned} X_1 &= V_1 \cos\theta_1 \\ Y_1 &= V_1 \sin\theta_1 \end{aligned} \tag{7.14}$$

$$\begin{aligned} X_1 &= \frac{w_1(a_1 - x)}{\sqrt{(x-a_1)^2 + (y-b_1)^2}} \\ Y_1 &= \frac{w_1(b_1 - y)}{\sqrt{(x-a_1)^2 + (y-b_1)^2}} \end{aligned} \tag{7.15}$$

The stationary position of the knot requires that the force components are balanced, i.e., sum up to zero.

$$\begin{aligned} \sum_j X_j &= 0 \\ \sum_j Y_j &= 0 \end{aligned} \tag{7.16}$$

$$\sum_j \frac{w_j\left(x-a_j\right)}{\sqrt{\left(x-a_j\right)^2+(y-b_j)^2}}=0$$
$$\sum_j \frac{w_j\left(y-b_j\right)}{\sqrt{\left(x-a_j\right)^2+(y-b_j)^2}}=0 \tag{7.17}$$

These forces at equilibrium equations are equal to the gradient optimality conditions for the Euclidean distance minisum problem.

$$z=\sum_j w_j\sqrt{\left(x-a_j\right)^2+(y-b_j)^2}$$
$$\frac{\partial z}{\partial x}=\sum_j \frac{w_j\left(x-a_j\right)}{\sqrt{\left(x-a_j\right)^2+(y-b_j)^2}}=0 \tag{7.18}$$
$$\frac{\partial z}{dy}=\sum_j \frac{w_j\left(y-b_j\right)}{\sqrt{\left(x-a_j\right)^2+(y-b_j)^2}}=0$$

Hyperboloid Approximation

To avoid the singularities of the derivative at the existing facilities, the distance norm can be perturbed by adding a small constant ε.

$$d^E(X_i,P_j)=\sqrt{(x_i-a_j)^2+(y_i-b_j)^2+\varepsilon} \tag{7.19}$$

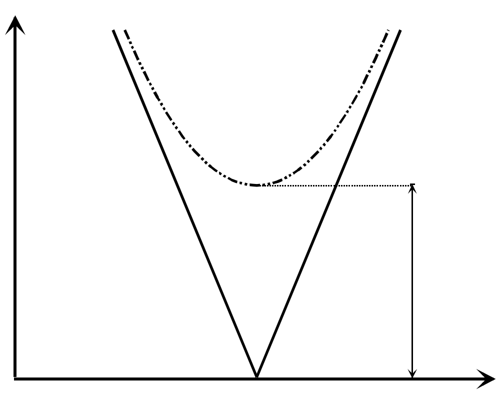


Figure 7.13. Illustration of the Hyperboloid Approximation

The partial derivatives for a single existing facility are then given in the next equation.

$$\frac{\partial d}{\partial x_i}=\frac{(x_i-a_j)}{\sqrt{(x_i-a_j)^2+(y_i-b_j)^2+\varepsilon}} \tag{7.20}$$

The partial derivatives for a multiple existing facilities are then given in the next equation.

$$\frac{\partial d}{\partial x_i} = \sum_j \frac{w_j (x_i - a_j)}{\sqrt{(x_i - a_j)^2 + (y_i - b_j)^2 + \varepsilon}} \tag{7.21}$$

Weiszfeld's Iterative Procedure

$$g_{ij}(x_i, y_i) = \frac{w_{ij}}{\sqrt{\left(x_i - a_j\right)^2 + \left(y_i - b_j\right)^2 + \varepsilon}} \tag{7.22}$$

$$\begin{aligned} \sum_j (x_i - a_j) \cdot g_{ij} &= 0 \\ \sum_j (y_i - b_j) \cdot g_{ij} &= 0 \end{aligned} \tag{7.23}$$

$$\begin{aligned} x_i \sum_j g_{ij} &= \sum_j a_j \cdot g_{ij} \\ y_i \sum_j g_{ij} &= \sum_j b_j \cdot g_{ij} \end{aligned} \tag{7.24}$$

$$\lambda_{ij}(x_i, y_i) = \frac{g_{ij}(x_i, y_i)}{\sum_j g_{ij}(x_i, y_i)} \tag{7.25}$$

$$\begin{aligned} &0 \le \lambda_{ij} \le 1 \\ &\sum_j \lambda_{ij} = 1 \end{aligned} \tag{7.26}$$

$$\begin{aligned} x_i &= \sum_j \lambda_{ij}(x_i, y_i) \cdot a_j \\ y_i &= \sum_j \lambda_{ij}(x_i, y_i) \cdot b_j \end{aligned} \tag{7.27}$$

$$\begin{cases} x_i^{k+1} = \sum_j \lambda_{ij}^k \left(x_i^k, y_i^k\right) \cdot a_j \\ y_i^{k+1} = \sum_j \lambda_{ij}^k \left(x_i^k, y_i^k\right) \cdot b_j \end{cases} \tag{7.28}$$

Properties

Convex Hull Property

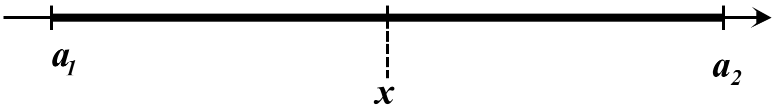


Figure 7.14. Convex Combination Illustration

$$\begin{aligned} & x = \lambda a_1 + (1-\lambda) a_2 \\ & 0 \le \lambda \le 1 \end{aligned} \qquad (7.29)$$

$$\begin{aligned} & x = \lambda_1 a_1 + \lambda_2 a_2 \\ & \lambda_1 + \lambda_2 = 1 \\ & 0 \le \lambda \le 1 \end{aligned} \qquad (7.30)$$

Majority Property (Scalar Sum)

A sufficient but not necessary condition for the location of an exiting facility *k* to be the optimal location of the Euclidean single facility minisum problem is

$$\left.\begin{aligned} & w_k \ge \frac{W}{2} = \frac{\sum_{j=1}^{N} w_j}{2} \\ & w_k \ge \sum_{j=1, j\ne k}^{N} w_j \end{aligned}\right\} \Rightarrow P_k = X^* \qquad (7.31)$$

This property can be interpreted based on the mechanical analog of the Varignon frame. An existing facility k is the optimal location if the vector sum of all the forces to the other existing facilities is smaller than the affinity with facility *k*, even if all the other existing facilities fall on a line through facility *k*. This property is only based on the size of the forces, not on their direction. Given the name of the next property, the majority property could then also be called the scalar sum property.

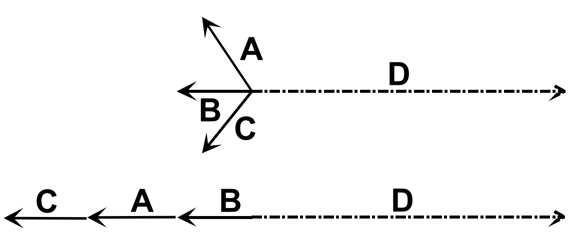


Figure 7.15. Scalar Sum Property Illustration

Vector-Sum Property

The location of an existing facility k is the optimal location of the Euclidean single facility minisum problem if and only if the following inequality holds

$$\sqrt{\left(\sum_{j=1,j\neq k}^{N}\frac{w_j(a_k-a_j)}{\sqrt{(a_k-a_j)^2+(b_k-b_j)^2}}\right)^2+\left(\sum_{j=1,j\neq k}^{N}\frac{w_j(b_k-b_j)}{\sqrt{(a_k-a_j)^2+(b_k-b_j)^2}}\right)^2}\leq w_k \Rightarrow P_k=X^* \tag{7.32}$$

This property can be interpreted based on the mechanical analog of the Varignon frame. An existing facility k is the optimal location if the vector sum of all the forces to the other existing facilities is smaller than the affinity with facility k.

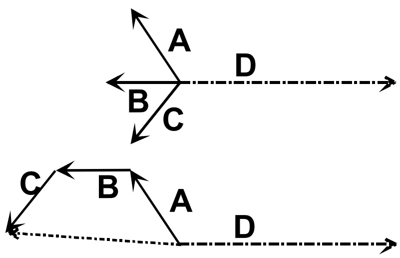


Figure 7.16. Vector Sum Property Illustration

This property allows checking all the existing facilities for optimality before entering the iterative procedure, which then never has to consider or visit the existing facilities, where its derivative is undefined.

Lower Bounds

Lower and Upper Bounds based on the Rectilinear Norm

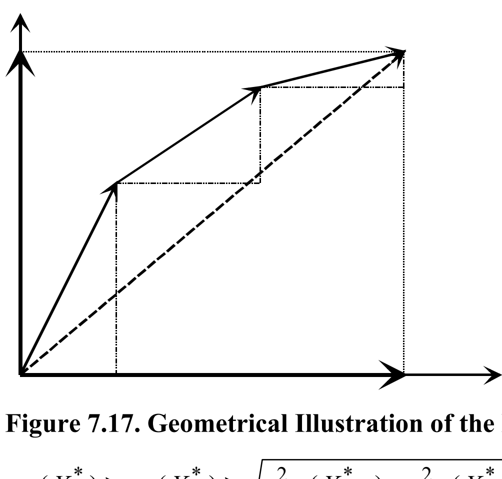


Figure 7.17. Geometrical Illustration of the Rectilinear Norm Based Lower Bound

$$z_R(X_R^*)\geq z_E(X_E^*)\geq\sqrt{z_{RX}^2(X_{RX}^*)+z_{RY}^2(X_{RY}^*)} \tag{7.33}$$

The disadvantage of this lower bound is that it requires the optimal solution to another location problem. Luckily, solving the rectilinear point location problem is easy. The advantage of this lower bound is that it does not require the determination of the vertex points of the convex hull of existing facilities like the lower bound derived below and that it needs only to be computed once.

Lower Bound based on Convex Function Support

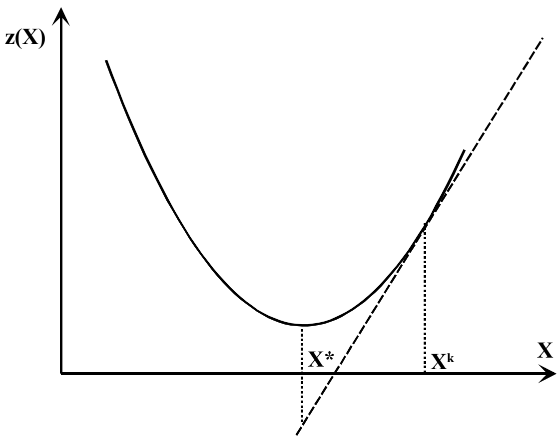


Figure 7.18. Convex Function Support

$$z(X) \geq z(X^k) + \nabla z(X^k) \cdot (X - X^k) \qquad \forall X \tag{7.34}$$

$$\begin{aligned} z(X^*) &\geq z(X^k) + \nabla z(X^k) \cdot (X^* - X^k) \\ &\geq z(X^k) - \left| \nabla z(X^k) \cdot (X^* - X^k) \right| \\ &\geq z(X^k) - \left\| \nabla z(X^k) \right\| \cdot \left\| X^* - X^k \right\| \\ &\geq z(X^k) - \left\| \nabla z(X^k) \right\| \cdot \max_j \left\| P_j - X^k \right\| \end{aligned} \tag{7.35}$$

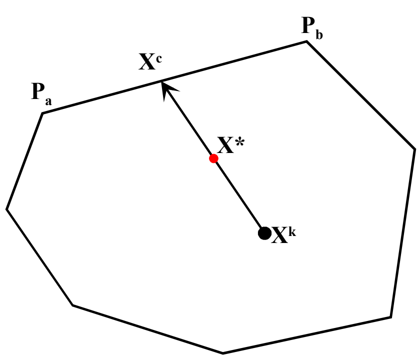


Figure 7.19. Vector Sizes inside the Convex Hull

$$\left\| X^* - X^k \right\| \leq \left\| X^c - X^k \right\| \leq \max\{ \left\| P_a - X^k \right\|, \left\| P_b - X^k \right\| \} \leq \max_j \left\| P_j - X^k \right\| \tag{7.36}$$

Location Algorithm

1. Problem Reduction

Majority Theorem

Vector Sum Theorem

2. Initial Starting Point

Center of Gravity

Largest Fixed Facility

Optimal Rectilinear Location

3. Check Stopping Criteria

Maximum Number of Iterations

Compute Lower Bound

Stop if Gap within Tolerance

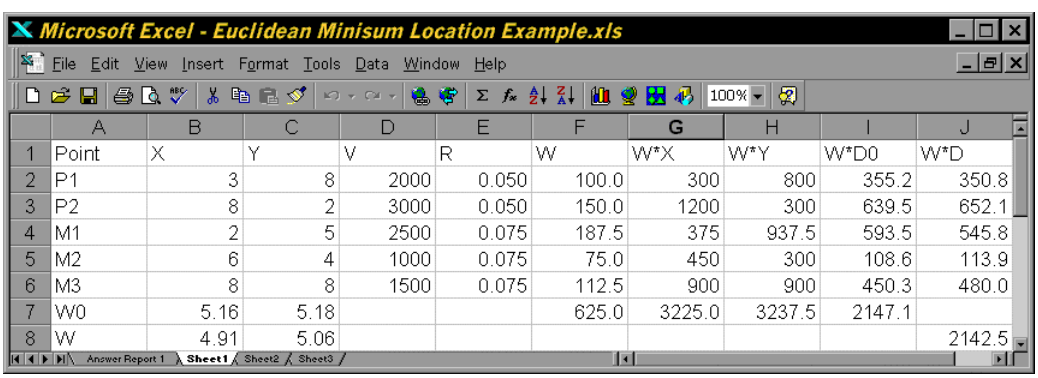
4. Compute Next Location

Weiszfeld's Method -

Hyperboloid Approximation

Go to Step 3

Single Facility Location Example



	A	B	C	D	E	F	G	H	I	J
1	Point	X	Y	V	R	W	W*X	W*Y	W*D0	W*D
2	P1	3	8	2000	0.050	100.0	300	800	355.2	350.8
3	P2	8	2	3000	0.050	150.0	1200	300	639.5	652.1
4	M1	2	5	2500	0.075	187.5	375	937.5	593.5	545.8
5	M2	6	4	1000	0.075	75.0	450	300	108.6	113.9
6	M3	8	8	1500	0.075	112.5	900	900	450.3	480.0
7	W0	5.16	5.18			625.0	3225.0	3237.5	2147.1	
8	W	4.91	5.06							2142.5

Figure 7.20. Excel Spreadsheet for Single Facility Location

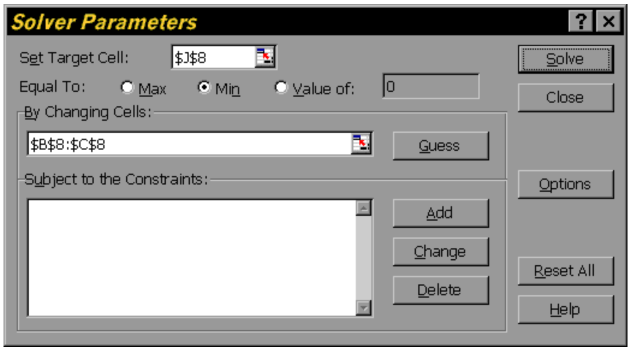


Figure 7.21. Excel Solver Parameters for Single Facility Location

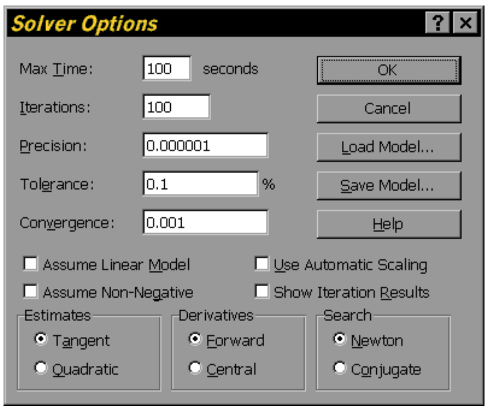


Figure 7.22. Excel Solver Options for Single Facility Location

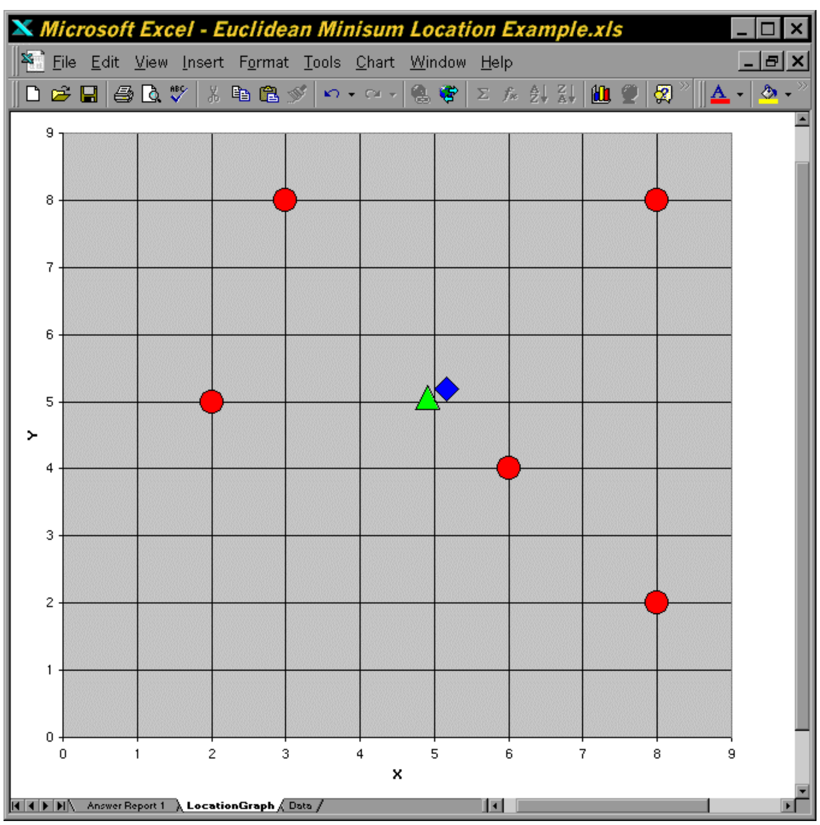


Figure 7.23. Excel Location Graph for Single Facility Location

Multiple Facility Minisum Location

H is the set of movable facilities and G is the set of fixed facilities. w_{ij} is the affinity or flow from moveable facility i to fixed facility *j*, v_{ij} is the affinity or flow from movable facility *i* to moveable facility *j*. (x_i, y_i) is the variable location of the moveable facility *i* and (a_j, b_j) is the location of the fixed facility *j*. The Euclidean minisum problem is then given by.

$$\begin{aligned} Min\ z = & \sum_{i=1}^{H}\sum_{j=1}^{G} w_{ij}\sqrt{\left(x_i - a_j\right)^2 + \left(y_i - b_j\right)^2} + \\ & \sum_{i=1}^{H}\sum_{j=i+1}^{H} v_{ij}\sqrt{\left(x_i - x_j\right)^2 + \left(y_i - y_j\right)^2} \end{aligned} \tag{7.37}$$

To minimize *z* the partial derivatives with respect to x and y are calculated and set to zero, which yields the following recursive expressions for x and y.

$$\begin{aligned} \frac{\partial z}{\partial x_i} &= \sum_{j}^{G} \frac{w_{ij}\left(x_i - a_j\right)}{\sqrt{\left(x_i - a_j\right)^2 + (y_i - b_j)^2}} + \sum_{j, j \neq i}^{H} \frac{v_{ij}\left(x_i - x_j\right)}{\sqrt{\left(x_i - x_j\right)^2 + (y_i - y_j)^2}} = 0 \\ \frac{\partial z}{d y_i} &= \sum_{j}^{G} \frac{w_{ij}\left(y_i - b_j\right)}{\sqrt{\left(x_i - a_j\right)^2 + (y_i - b_j)^2}} + \sum_{j, j \neq i}^{H} \frac{v_{ij}\left(y_i - y_j\right)}{\sqrt{\left(x_i - x_j\right)^2 + (y_i - y_j)^2}} = 0 \end{aligned} \tag{7.38}$$

First, define the following two functions:

$$g_{ij}(x_i, y_i) = \frac{w_{ij}}{\sqrt{\left(x_i - a_j\right)^2 + \left(y_i - b_j\right)^2 + \varepsilon}} \tag{7.39}$$

$$\begin{aligned} h_{ij}(x_i, y_i) &= \frac{v_{ij}}{\sqrt{\left(x_i - x_j\right)^2 + \left(y_i - y_j\right)^2 + \varepsilon}} \qquad i \neq j \\ h_{ii}(x_i, y_i) &= 0 \end{aligned} \tag{7.40}$$

Note that the denominators are adjusted by the (small) positive constant ε. This prevents the denominators from ever being zero. Without the ε term, these functions would be undefined whenever a moveable facility was located at the same site as a fixed facility. This adjustment method is called the Hyperbolic Approximation Procedure or HAP. Further details can be found in Francis and White (1974) and Love et al. (1988).

Now, the set of locations $\left(x^k, y^k\right)$ is determined as follows:

$$x_i^k = \frac{\sum_{j=1}^{G} a_j g_{ij}\left(x_i^{k-1}, y_i^{k-1}\right) + \sum_{j=1}^{H} x_j h_{ij}\left(x_i^{k-1}, y_i^{k-1}\right)}{\sum_{j=1}^{G} g_{ij}\left(x_i^{k-1}, y_i^{k-1}\right) + \sum_{j=1}^{H} h_{ij}\left(x_i^{k-1}, y_i^{k-1}\right)} \tag{7.41}$$

$$y_i^k = \frac{\sum_{j=1}^{G} b_j g_{ij}\left(x_i^{k-1}, y_i^{k-1}\right) + \sum_{j=1}^{H} y_j h_{ij}\left(x_i^{k-1}, y_i^{k-1}\right)}{\sum_{j=1}^{G} g_{ij}\left(x_i^{k-1}, y_i^{k-1}\right) + \sum_{j=1}^{H} h_{ij}\left(x_i^{k-1}, y_i^{k-1}\right)} \tag{7.42}$$

The recursive formulas require an initial location $\left(x^0, y^0\right)$ as input and then the procedure uses an iterative improvement scheme to get the next estimation. The superscript k denotes the iteration number. The iterative procedure continues until a stopping criterion is satisfied. Examples of stopping criteria are the location of all moveable facilities falls within the relative tolerance of their previous location, the objective function is within an acceptable tolerance of a lower bound on the objective function, or the maximum number of iterations has been reached.

Minisum Location-Allocation

Introduction

The strategic logistics design problem can be defined as follows: given a set of plants and customers with known characteristics, and the potential components of a logistics network, determine the number and location of warehouses, allocate of customers to warehouses, and select transportation channels such that customer requirements are met at the lowest possible cost.

Solution procedures for the strategic logistics system design problem can be divided into two types, each with different assumptions. Site generating procedures, such as location-allocation solution procedures, generate a set of new sites for the distribution centers, but do not consider whether to open or close distribution centers. That is, location-allocation procedures assume that each specified potential distribution center is open (though not necessarily used). Thus, to minimize total relevant costs, location-allocation procedures need only minimize the variable cost, which include at least the total transportation cost. To summarize, the location-allocation procedures minimize distribution cost by moving the distribution centers, while leaving their number unchanged.

The second class of solution procedures selects desirable distribution centers to open from among a list of possible candidate locations. They are called site selection procedures. The optimal solution algorithms of this type are based on Mixed Integer Programming (MIP) techniques. The MIP procedures determine the optimal number of distribution centers to open out of a set of candidate distribution centers. The candidate distribution centers are fixed in place. A mathematical model is constructed which captures all the cost and the solution is obtained by using a MIP solution program such as CPLEX, LINDO or MINTO. To summarize, the MIP procedures minimize the distribution cost by opening and closing distribution centers, while leaving their location unchanged. These procedures are further discussed in the chapter on discrete location.

Eilon-Watson-Ghandi Iterative Location-Allocation Algorithm

This particular location-allocation procedures attempts to find a good set of locations for the distribution centers by repeatedly executing the following steps:

1. Allocate customers to depots and depots to plants
2. Relocate depots to minimize transportation costs of the current allocations by using .
3. Repeat steps 1 and 2 until done.

The allocation phase is solved via a network flow algorithm. The location phase is solved by the Weiszfeld algorithm with hyperbolic approximation. Further details are given below.

The network flow model considers plant, customers and depots. It determines the location of the distribution centers and the allocation of customers to distribution centers based on transportation costs only. The distribution centers are capacitated and flows between the distribution centers are allowed. In addition, more than one flow between two facilities is allowed. This corresponds to having more than one transportation mode between two facilities.

The following iterative and heuristic algorithm obtains the solution.

Allocation Phase

The algorithm starts with an initial solution in which the initial location of the distribution centers is specified. This initial location can be random, specified by the user, or the result of another algorithm.

Based on this initial location, the network flow algorithm computes transportation costs and then assigns each customer to the nearest distribution center or plant with sufficient capacity by solving the following network flow problem:

$$\text{Min.}\sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{m=1}^{L} c_{ijm} d_{ijm} w_{ijm} \qquad (7.4$$

$$\text{s.t.}\sum_{i=1}^{M}\sum_{m=1}^{L} w_{ijm} = dem_k \;\; k \in Customers \qquad (7.44)$$

$$\sum_{j=1}^{N}\sum_{m=1}^{L} w_{ijm} \le cap_i \;\; i \in Plants \qquad (7.45)$$

$$\sum_{i=1}^{M}\sum_{m=1}^{L} w_{ijm} - \sum_{k=1}^{N}\sum_{m=1}^{L} w_{jkm} = 0 \;\; j \in Depots \qquad (7.46)$$

$$w_{ijm} \ge 0$$

M is the total number of source facilities which is equal to the number of plants plus the number of depots. N is the total number of sink facilities which is equal to the number of depots plus the number of customers. L is the number of transportation modes. The modes are indexed by the subscript m. w_{ijm} is the flow from facility i to facility j by mode m, c_{ijm} is the cost per unit flow per unit distance for transportation from facility i to facility j by mode m, d_{ijm} is the distance from facility i to facility j by mode m, dem_k is the demand of customer k, and cap_i is the capacity of plant i. Further information on solving the network flow problem can be found in Bazaraa and Jarvis (1977).

Location Phase

After all the customers have been allocated to the nearest distribution center with available capacity, a second sub-algorithm locates the distribution centers so that the sum of the weighted distances between each source and sink facility is minimized for the given flows. This problem is formulated as a continuous, multiple facility weighted Euclidean minisum location problem:

$$\begin{aligned} Min\ f(x,y) = \quad & \sum_{i=1}^{H}\sum_{j=1}^{G}\sum_{m=1}^{L} c_{ijm} w_{ijm} \sqrt{\left(x_i - a_j\right)^2 + \left(y_i - b_j\right)^2} + \\ & \sum_{i=1}^{H}\sum_{j=1}^{G}\sum_{m=1}^{L} c_{ijm} v_{ijm} \sqrt{\left(x_i - x_j\right)^2 + \left(y_i - y_j\right)^2} \end{aligned} \qquad (7.47)$$

A preprocessing step combines all modes (L) between a pair of facilities into one aggregate mode:

$$Min\ f(x,y) = \sum_{i=1}^{H}\sum_{j=1}^{G} c_{ij} w_{ij} \sqrt{\left(x_i - a_j\right)^2 + \left(y_i - b_j\right)^2} + \sum_{i=1}^{H}\sum_{j=1}^{H} c_{ij} v_{ij} \sqrt{\left(x_i - x_j\right)^2 + \left(y_i - y_j\right)^2} \tag{7.48}$$

H is the set of movable facilities, i.e. the distribution centers. G is the set of fixed facilities which is composed of plants and customers. w_{ij} is the flow from moveable facility i to fixed facility j, v_{ij} is the flow from movable facility i to moveable facility j. (x_i, y_i) is the variable location of the distribution center i and $\left(a_j, b_j\right)$ is the fixed location of customer j or of plant j. The distance between centers is assumed to be proportional to the straight line Euclidean distance.

To minimize $f(x,y)$ the partial derivatives with respect to x and y are calculated and set to zero which yields the following recursive expressions for *x* and *y*. First, define the following two functions:

$$g_{ij}(x_i, y_i) = \frac{c_{ij} w_{ij}}{\sqrt{\left(x_i - a_j\right)^2 + \left(y_i - b_j\right)^2 + \varepsilon}} \tag{7.49}$$

$$h_{ij}(x_i, y_i) = \frac{c_{ij} v_{ij}}{\sqrt{\left(x_i - x_j\right)^2 + \left(y_i - y_j\right)^2 + \varepsilon}} \qquad i \neq j$$
$$h_{ii}(x_i, y_i) = 0 \tag{7.50}$$

Note that the denominators are adjusted by the (small) positive constant ε. This prevents the denominators from ever being zero. Without the ε term, these functions would be undefined whenever a distribution center was located at the same site as a customer. This adjustment method is called the Hyperbolic Approximation Procedure or HAP. Further details can be found in Francis and White (1974) and Love et al. (1988).

Now, the set of locations $\left(x^k, y^k\right)$ is determined as follows:

$$x_i^k = \frac{\sum_{j=1}^{G} a_j g_{ij}\left(x_i^{k-1}, y_i^{k-1}\right) + \sum_{j=1}^{H} x_j h_{ij}\left(x_i^{k-1}, y_i^{k-1}\right)}{\sum_{j=1}^{G} g_{ij}\left(x_i^{k-1}, y_i^{k-1}\right) + \sum_{j=1}^{H} h_{ij}\left(x_i^{k-1}, y_i^{k-1}\right)} \tag{7.51}$$

$$y_i^k = \frac{\sum_{j=1}^{G} b_j g_{ij}\left(x_i^{k-1}, y_i^{k-1}\right) + \sum_{j=1}^{H} y_j h_{ij}\left(x_i^{k-1}, y_i^{k-1}\right)}{\sum_{j=1}^{G} g_{ij}\left(x_i^{k-1}, y_i^{k-1}\right) + \sum_{j=1}^{H} h_{ij}\left(x_i^{k-1}, y_i^{k-1}\right)} \tag{7.52}$$

The recursive formulas require an initial location $\left(x^0, y^0\right)$ as input and then the procedure uses an iterative improvement scheme to get the next estimation. The index k denotes the iteration number. The iterative procedure continues until a stopping criterion has been satisfied.

Single Facility Minimax Location

Assumptions

Single New Facility

Equal Affinities

Primal Algorithm

1. Pick Initial Radius z0, set k = 0
2. Draw Circles Around Pj with radii zk
3. Determine Intersection Z of the Circles
4. If Z is Empty, Increase zk, k=k+1, Go to 2
   If Z is More than Single Point, Decrease zk, k=k+1, Go to 2
   If Z is Single Point, Stop

Dual Elzinga-Hearn Algorithm

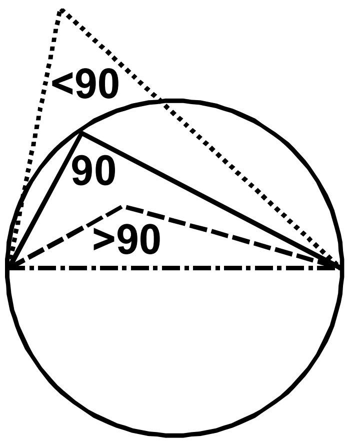


Figure 7.24. Angles of Triangles based on Circle Perimeters

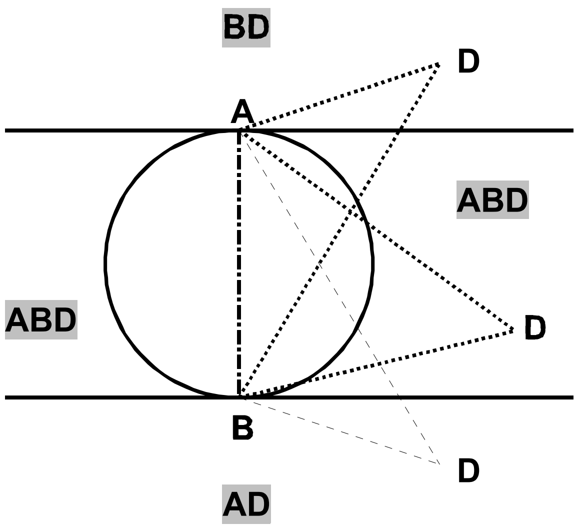


Figure 7.25. Regions for Extending a Circle based on Two Points

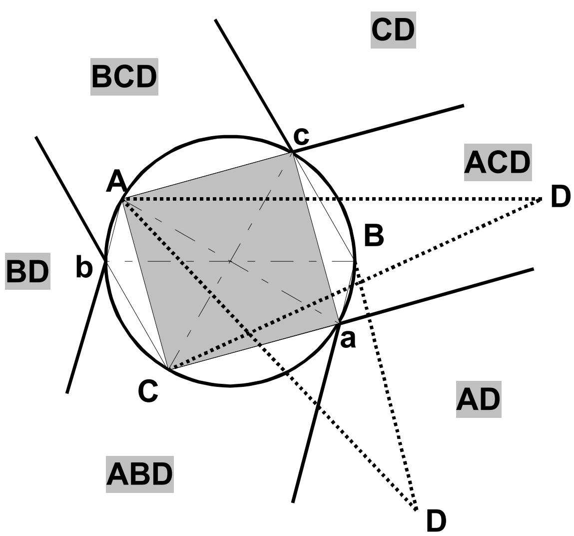


Figure 7.26. Regions for Extending a Circle based on Three Points

Rectilinear Location

Rectilinear Distance Norm

Norm

$$d_R = \left(\left| x - a_j \right| + \left| y - b_j \right| \right) \tag{7.53}$$

Applications

Factories and warehouse with aisles and cross aisles.

Cities with a street plan of perpendicular streets and avenues such as the prototypical Manhattan.

Material handling equipment with sequential travel.

Used in lower bound computation for the Euclidean minisum location and location-allocation algorithms.

Graph

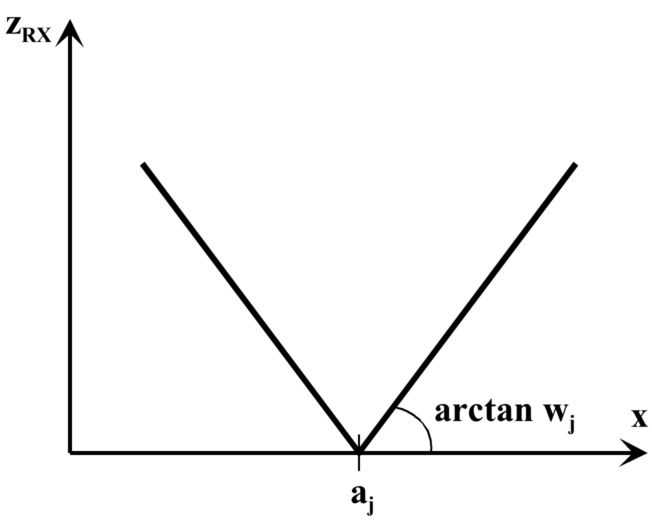


Figure 7.27. Rectilinear Distance Norm Component Graph

Properties

Distance Norm Properties

Continuous

Convex

Non-Differentiable

Decomposable in independent X and Y components

Single Facility Minisum Location

We assume that all weights are nonnegative.

$$w_j \geq 0 \qquad \forall j \tag{7.5}$$

Objective Function

$$\min Z_R = \sum_{j=1}^{N} w_j \left(\left| x - a_j \right| + \left| y - b_j \right| \right) \tag{7.56}$$

$$\begin{aligned} \min Z_{RX}(X) &= \sum_{j=1}^{N} w_j \left| x - a_j \right| \\ \min Z_{RY}(X) &= \sum_{j=1}^{N} w_j \left| y - b_j \right| \end{aligned} \tag{7.57}$$

Vector Notation

$$X = \begin{bmatrix} x \\ y \end{bmatrix} \tag{7.58}$$

$$P_j = \begin{bmatrix} a_j \\ b_j \end{bmatrix} \tag{7.59}$$

$$W = \begin{bmatrix} w_1 \\ \ldots \\ w_N \end{bmatrix} \tag{7.60}$$

$$D_R = \begin{bmatrix} d_r(X, P_1) \\ \ldots \\ d_r(X, P_N) \end{bmatrix} \tag{7.61}$$

$$Z_R(X) = \sum_{j=1}^{N} w_j d_R(X, P_j) = W^T D_R \tag{7.62}$$

Majority Property or Witzgall Property

$$w_k \geq \frac{W}{2} = \frac{\sum_{j=1}^{N} w_j}{2} \Rightarrow P_k = X^* \tag{7.63}$$

Minisum Objective Function Graph

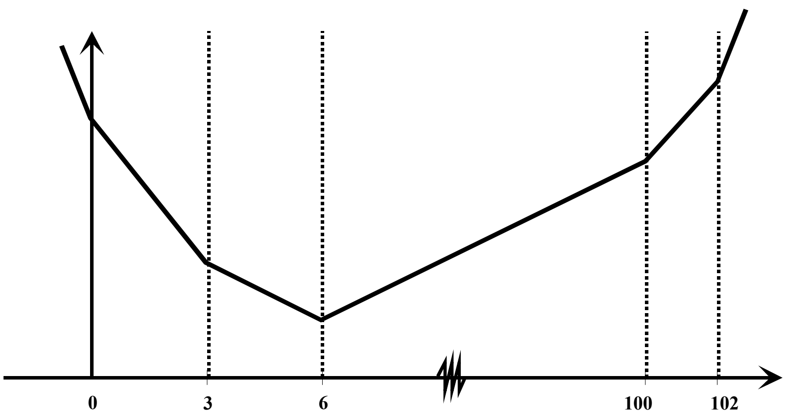


Figure 7.28. Rectilinear Minisum Objective Function Illustration

Table 7.1. Rectilinear Minisum Function Characteristics

a_j		0		3		6		100		102	
w_j		5		1		3		2		4	
L_j	0		5		6		9		11		15
R_j	15		10		9		6		4		0
s_x	-15		-5		-3		3		7		15

Properties of the Objective Function and Optimal Location

Piecewise linear objective function.

Breakpoints occur at the coordinates of the existing facilities, where the slop changes.

The optimal location falls at the location of an existing facility or alternative optima between the location of two existing facilities.

Since the objective function is convex, this is a global optimum.

Median Conditions

Renumber existing facilities by increasing coordinates along the axis. The optimal location is then where the sum of the weights of the facilities to the left is larger than or equal to half the total weight.

$$W = \sum_{j=1}^{N} w_j \tag{7.64}$$

$$j^* \Leftarrow \left\{ \sum_{j=1}^{j^*-1} w_j < \frac{W}{2}, \sum_{j=1}^{j^*} w_j \geq \frac{W}{2} \right\} \tag{7.65}$$

Grid Lines

Grid lines are the horizontal and vertical lines through the existing facilities.

Optimal location falls on the intersection of two grid lines. There can exist alternative locations on a line segment of a grid line determined by the intersection of two adjacent grid lines. The set of alternative locations can also be a rectangle determined by two pairs of adjacent grid lines that intersect.

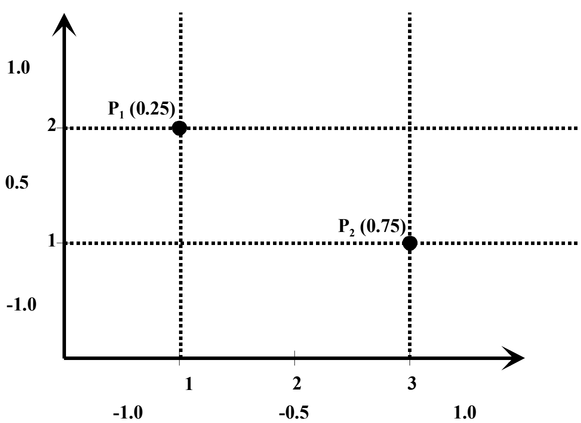


Figure 7.29. Grid Lines Illustration

Contour Lines

Contour lines are lines of equal objective function value. They are also called iso-cost lines. Inside a rectangle bounded by two pair off adjacent contour lines. The slope of a contour line can be found based on the following formula.

$$\Delta x \cdot s_x + \Delta y \cdot s_y = 0 \tag{7.66}$$

$$s_c = \frac{\Delta y}{\Delta x} = -\frac{s_x}{s_y} \tag{7.67}$$

The slope of the contour curve is constant inside the box, so the curve is a line segment. All the contour curves in a box of the grid are parallel lines. The contour lines themselves are closed polygons.

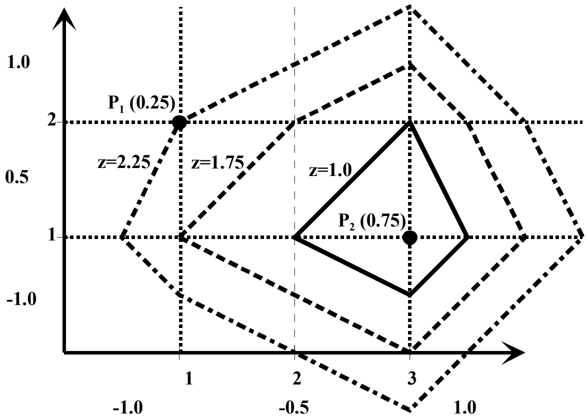


Figure 7.30. Contour Lines Illustration

Linear Programming Formulation

Recall that the objective function for the single facility minisum location problem is based on the rectilinear norm is given by:

$$\min Z_R = \sum_{j=1}^{N} w_j \left(\left| x - a_j \right| + \left| y - b_j \right| \right) \tag{7.68}$$

This primal objective decomposes into two independent objectives functions based exclusively upon x and y, and denoted by Z_{RX} and Z_{RY}, respectively. We will restrict ourselves to Z_{RX} in the following discussion since the equivalent steps can be executed for the y component of the problem. The objective functions for the horizontal and vertical subproblems are then

$$\min Z_{RX}(X) = \sum_{j=1}^{N} w_j \left| x - a_j \right| \tag{7.69}$$

$$\min Z_{RY}(X) = \sum_{j=1}^{N} w_j \left| y - b_j \right| \tag{7.70}$$

To circumvent the problem of the non-linear absolute value operator two auxiliary variables p_j^+ and p_j^- are introduced.

$$\begin{cases} p_j^+ = \left| x - a_j \right| & \text{if } x \geq a_j, 0 \text{ otherwise} \\ p_j^- = \left| x - a_j \right| & \text{if } x < a_j, 0 \text{ otherwise} \end{cases} \tag{7.71}$$

This yields the following equations.

$$\begin{aligned} &\left|x-a_j\right| = p_j^+ + p_j^- \\ &x - p_j^+ + p_j^- = a_j \qquad \text{if } p_j^+ \cdot p_j^- = 0 \end{aligned} \tag{7.72}$$

Using these equations, the primal formulation is transformed from an unconstrained nonlinear optimization problem to a linear constrained optimization formulation.

$$\begin{aligned} \min \quad & \sum_{j=1}^{N} w_j\left(p_j^+ + p_j^-\right) \\ s.t. \quad & x - p_j^+ + p_j^- = a_j \qquad \forall j \qquad [u_j] \\ & x \ \ unrestricted \\ & p \geq 0 \end{aligned} \tag{7.73}$$

Observe that the simplex method for linear programming automatically satisfies the conditions that one of the pair of auxiliary variables must be zero, so this constraint need not be explicitly included in the formulation. p_j^+ and p_j^- cannot be simultaneously significant, i.e., positive, in the optimal solution of the primal formulation since their coefficient columns are linearly dependent and the basic, non-zero variables must have linearly independent columns. This can also be seen by example. Assume that $p_j^+ = 3$ and $p_j^- = 1$, then the constraint reduces to $x - 2 = a_j$ and the objective function value is $4w_j$. If both p_j's are reduced by one, then $p_j^+ = 2$ and $p_j^- = 0$. The constraint reduces again to $x - 2 = a_j$ and the objective function value is $2w_j$, which is clearly better. Observe that the above discussion requires that all weights $w_j \geq 0$, otherwise the problem becomes unbounded if this transformation is applied.

Note that additional linear constraints can be added without making the problem significantly harder as long the constraints also decompose into independent x and y components. Adding constraints in both x and y will destroy the decomposition property and instead of two smaller linear programming formulation a single formulation of twice the size has to be solved.

The dual formulation is given by

$$
\begin{aligned}
\max \quad & \sum_{j=1}^{N} a_j u_j \\
s.t. \quad & \sum_{j=1}^{N} u_j = 0 & & [x] \\
& -w_j \le u_j & \forall j \quad & [p_j^+] \\
& u_j \le w_j & \forall j \quad & [p_j^-] \\
& u_j \ \textit{unrestricted}
\end{aligned}
\tag{7.74}
$$

Using the following variable and parameter transformations the dual can be condensed and converted to a minimization problem.

$$
\begin{aligned}
& r_j = w_j - u_j \qquad \forall j \\
& \sum_{j=1}^{N} w_j = f
\end{aligned}
\qquad (7.7
$$

$$
\begin{aligned}
\min \quad & \sum_{j=1}^{N} a_j r_j \\
s.t. \quad & \sum_{j} r_j = f \\
& 0 \le r_j \le 2w_j \qquad \forall j
\end{aligned}
\tag{7.76}
$$

The following redundant (linearly dependent) constraint can be created and be added to the dual.

$$
-\sum_{j=1}^{N} r_j = -f \tag{7.77}
$$

$$
\begin{aligned}
\min \quad & \sum_{j=1}^{N} a_j r_j \\
s.t. \quad & \sum_{j} r_j = f \\
& -\sum_{j} r_j = -f \\
& 0 \le r_j \le 2w_j \qquad \forall j
\end{aligned}
\tag{7.78}
$$

This linear programming formulation corresponds to a network flow problem with a node corresponding to the new facility and a sink node, since all variables *r* have exactly two non-zero coefficients, equal to +1 and -1, respectively. The first two constraints represent the flow balance conditions in the new facility and sink nodes and the last set represents the lower and upper bounds constraints on the flows.

This network is a minimum cost network and, more in particular, a transportation network, which can be solved with a variety of network flow algorithms. The network structure is illustrated in Figure 7.31.

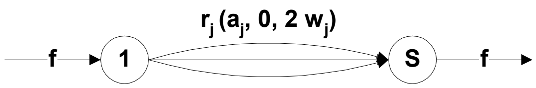


Figure 7.31. Single Facility Transportation Network

$$f = \sum_{j=1}^{N} w_j$$
$$j^* \Leftarrow \left\{ \sum_{j=1}^{j^*-1} 2w_j < f, \sum_{j=1}^{j^*} 2w_j \geq f \right\} \tag{7.79}$$

This simple transportation network can be solved optimally by the following greedy algorithm.

Algorithm 7.1. Single Facility Transportation Network

1. Order and renumber existing facilities
   rank the existing facilities j by non-decreasing a_j and renumber them by this sequence

2. Initialize the algorithm
   set j = 1, set F = f

3. Assign the flows sequentially
   if $2w_j \geq F$

   then set $r_j = F$, $j^* = j$, and stop

   else set $r_j = 2w_j$, $F = F - 2w_j$, $j = j + 1$, and go to step 3

The primal optimal solution to the original location problem can be found by using the complementary slackness conditions and the dual variables at optimality. The stopping existing facility j^* is determined by the condition that $0 < r_{j^*} = F < 2w_{j^*}$ or $0 < w_{j^*} - u_{j^*} < 2w_{j^*}$. This means that these dual constraints are not binding and the complementary slackness conditions

$$\begin{cases} (-w_{j^*} - u_{j^*}) \cdot p_{j^*}^{+} = 0 \\ (w_{j^*} - u_{j^*}) \cdot p_{j^*}^{-} = 0 \end{cases} \tag{7.80}$$

force $p_{j^*}^{+} = 0$ and $p_{j^*}^{-} = 0$, and thus

$$x^* - p_{j^*}^+ + p_{j^*}^- = a_{j^*} \text{ or}$$

$$x^* = a_{j^*}$$

In the case when $r_{j^*} = F$, $p_{j^*}^+$ and $p_{j^*+1}^-$ do not have to be zero and there exist alternative primal optimal locations between j^* and j^*+1.

Notice the similarities between this network solution method based on linear programming and the previous graphical algorithm based directly on the piecewise linear concave objective function. There the median conditions determined the optimal primal solution as the first existing facility for which the slope of the objective function became positive. Passing an existing facility going from left to right increased the objective function by $2w_j$.

Multiple Facility Minisum Location

Linear Programming Formulation

Objective *Function*

Just as in the case of the single new facility, the primal formulation decomposes again into independent x and y components. Let v_{ik} be the non-negative relationship between new (to be located) facilities i and k. The objective function for the horizontal subproblem is then

$$\min Z_{RX}(X) = \sum_{i=1}^{M} \sum_{j=1}^{N} w_{ij} \left| x_i - a_j \right| + \sum_{i=1}^{M-1} \sum_{k>i}^{M} v_{ik} \left| x_i - x_k \right| \tag{7.81}$$

The auxiliary variables p_{ij}^+ and p_{ij}^- are introduced as in the case of the single new facility. We define the auxiliary variables q_{ik}^+ and q_{ik}^- to represent the distance between the new facilities as:

$$\begin{cases} q_{ik}^+ = \left| x_i - x_k \right| & \text{if } x_i \geq x_k, 0 \text{ otherwise} \\ q_{ik}^- = \left| x_i - x_k \right| & \text{if } x_i < x_k, 0 \text{ otherwise} \end{cases} \tag{7.82}$$

This yields the following equations.

$$
\begin{aligned}
&\left|x_i - a_j\right| = p_{ij}^{+} + p_{ij}^{-} \\
&x_i - p_{ij}^{+} + p_{ij}^{-} = a_j \qquad if\ p_{ij}^{+} \cdot p_{ij}^{-} = 0 \\
&\left|x_i - x_k\right| = q_{ik}^{+} + q_{ik}^{-} \\
&x_i - x_k - q_{ik}^{+} + q_{ik}^{-} = 0 \qquad if\ q_{ik}^{+} \cdot q_{ik}^{-} = 0
\end{aligned} \tag{7.83}
$$

The problem again is transformed from an unconstrained nonlinear optimization problem to a linear constrained optimization formulation using these equations. Observe that again the simplex method for linear programming automatically satisfies the conditions that one of each pair of auxiliary variables must be zero.

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{M}\sum_{j=1}^{N} w_{ij}\left(p_{ij}^{+} + p_{ij}^{-}\right) + \sum_{i=1}^{M-1}\sum_{k>i}^{M} v_{ik}\left(q_{ik}^{+} + q_{ik}^{-}\right) \\
s.t. \quad & x_i - p_{ij}^{+} + p_{ij}^{-} = a_j \qquad \forall i, \forall j \qquad \left[u_{ij}\right] \\
& x_i - x_k - q_{ik}^{+} + q_{ik}^{-} = 0 \qquad \forall i, \forall k > i \qquad \left[l_{ik}\right] \\
& x \qquad unrestricted \\
& p \geq 0,\ q \geq 0
\end{aligned} \tag{7.84}
$$

The dual formulation is given by

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{M}\sum_{j=1}^{N} a_j u_{ij} \\
s.t. \quad & \sum_{j=1}^{N} u_{ij} + \sum_{k=i+1}^{M} l_{ik} - \sum_{k=1}^{i-1} l_{ki} = 0 \qquad \forall i \qquad \left[x_i\right] \\
& -w_{ij} \leq u_{ij} \qquad \forall i, \forall j \qquad \left[p_{ij}^{+}\right] \\
& u_{ij} \leq w_{ij} \qquad \forall i, \forall j \qquad \left[p_{ij}^{-}\right] \\
& -v_{ik} \leq l_{ik} \qquad \forall i, \forall k > i \qquad \left[q_{ik}^{+}\right] \\
& v_{ik} \leq l_{ik} \qquad \forall i, \forall k > i \qquad \left[q_{ik}^{-}\right] \\
& u, l \ \ unrestricted
\end{aligned} \tag{7.85}
$$

$$
\begin{array}{llll}
\max & \sum_{i=1}^{M}\sum_{j=1}^{N} a_j u_{ij} & & \\
s.t. & \sum_{j=1}^{N} u_{ij} + \sum_{k=i+1}^{M} l_{ik} - \sum_{k=1}^{i-1} l_{ki} = 0 & \forall i & [x_i] \\
 & -w_{ij} \le u_{ij} & \forall i, \forall j & [p_{ij}^{+}] \\
 & u_{ij} \le w_{ij} & \forall i, \forall j & [p_{ij}^{-}] \\
 & -v_{ik} \le l_{ik} & \forall i, \forall k > i & [q_{ik}^{+}] \\
 & v_{ik} \le l_{ik} & \forall i, \forall k > i & [q_{ik}^{-}] \\
 & u, l \ \ unrestricted & &
\end{array}
\tag{7.86}
$$

Using the following variable and parameter transformations the dual can be condensed and converted to a minimization problem.

$$
\begin{array}{ll}
r_{ij} = w_{ij} - u_{ij} & \forall i, \forall j \\
s_{ik} = v_{ik} - l_{ik} & \forall i, \forall k > i \\
\sum_{j} w_{ij} + \sum_{k>i} v_{ik} - \sum_{k<i} v_{ki} = f_i & \forall i
\end{array}
\tag{7.87}
$$

By adding the first set of constraints for all new facilities, denoted by i, the following redundant (linearly dependent) constraint can be created and be added to the dual.

$$
\sum_{i=1}^{M}\sum_{j=1}^{N} r_{ij} = \sum_{i=1}^{M} f_i = F \tag{7.8}
$$

$$
\begin{array}{lll}
\min & \sum_{i=1}^{M}\sum_{j=1}^{N} a_j r_{ij} & \\
s.t. & \sum_{j} r_{ij} + \sum_{k>i} s_{ik} - \sum_{k<i} s_{ki} = f_i & \forall i \\
 & 0 \le r_{ij} \le 2w_{ij} & \forall i, \forall j \\
 & 0 \le s_{ik} \le 2v_{ik} & \forall i, \forall k > i
\end{array}
\tag{7.89}
$$

$$
\begin{array}{lll}
\min & \sum_{i=1}^{M}\sum_{j=1}^{N} a_j r_{ij} & \\
s.t. & \sum_{j} r_{ij} + \sum_{k>i} s_{ik} - \sum_{k<i} s_{ki} = f_i & \forall i \\
 & \sum_{i=1}^{M}\sum_{j=1}^{N} r_{ij} = F & \\
 & 0 \le r_{ij} \le 2w_{ij} & \forall i, \forall j \\
 & 0 \le s_{ik} \le 2v_{ik} & \forall i, \forall k > i
\end{array}
\tag{7.90}
$$

This linear programming formulation again corresponds to a network flow problem with a node corresponding to every new facility and a single sink node, since all variables *r* and *s* have exactly two non-zero coefficients, equal to +1 and -1, respectively. The first two sets of constraints represent the flow balance conditions in every node and the next two sets are lower and upper bounds constraints on the flows. This network is a minimum cost network, which can be solved with a variety of network flow algorithms. The network structure is illustrated in Figure 7.32. The primal optimal solution to the original location problem can again be found by using the complementary slackness conditions and the dual variables at optimality.

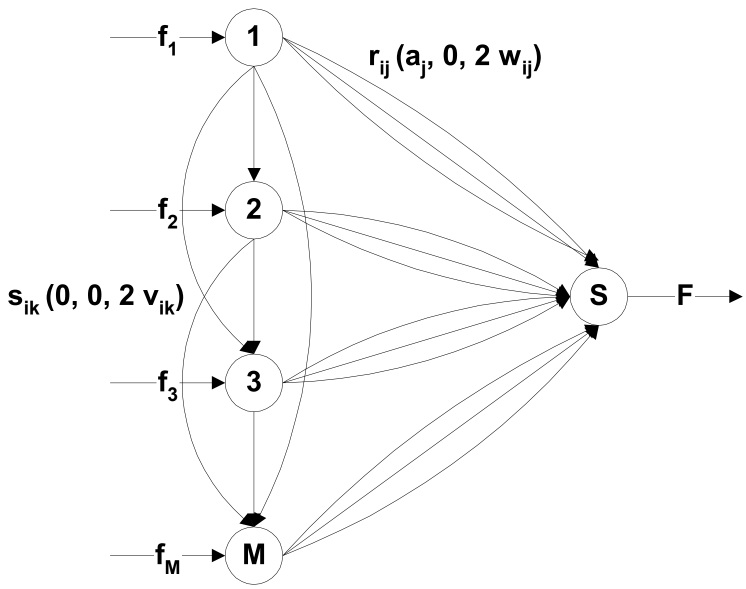


Figure 7.32. Multifacility Minimum Cost Network

Sequential Multifacility Algorithm

The sequential multifacility algorithm was developed by Picard and Ratliff (1978) and exploits the fact that the optimal location is based on the sequence and weights of the existing facilities, but not the interfacility distances. The optimal location is based on cuts in a sequence of q-locate networks.

Relationships between Euclidean and Rectilinear Location Problems

Upper and Lower Bound Property

Prove for the case of the multifacility location problem when there are relationships between the new facilities that the following series of inequalities holds:

$$z_R(X_R^*) \geq z_E(X_E^*) \geq \sqrt{z_{RX}^2(X_{RX}^*) + z_{RY}^2(X_{RY}^*)} \tag{7.91}$$

where E denotes the Euclidean norm, R denotes the rectilinear norm, and RX and RY denote the horizontal and vertical component of the rectilinear norm, respectively. X_E^* denotes the optimal Euclidean point and X_R^* denotes the optimal rectilinear point with horizontal and vertical components X_{RX}^* and X_{RY}^*, respectively.

Minimax Location

Graphical Algorithm

Algorithm 7.2. Rectilinear Minimax Location Graphical Algorithm

1. Draw Smallest Enclosing 45-Degree Rectangle of Existing Points
2. Extend the rectangle in the direction of the smallest side to Form a Diamond (Center of the Diamond is Point A)
3. Extend Rectangle in Opposite Direction to Form a Diamond (Center of the Diamond is Point B)
4. Line Segment AB is Set of Optimal Locations

Algebraic Solution

$$
\begin{aligned}
c_1 &= \min_j \{a_j + b_j\} \\
c_2 &= \max_j \{a_j + b_j\} \\
c_3 &= \min_j \{-a_j + b_j\} \\
c_4 &= \max_j \{-a_j + b_j\} \\
c_5 &= \max\{c_2 - c_1, c_4 - c_3\} \\
X^* &= \lambda \begin{bmatrix} \frac{c_1 - c_3}{2} \\ \frac{c_1 + c_3 + c_5}{2} \end{bmatrix} + (1-\lambda) \begin{bmatrix} \frac{c_2 - c_4}{2} \\ \frac{c_2 + c_4 - c_5}{2} \end{bmatrix}
\end{aligned}
\tag{7.92}
$$

Primal Algorithm

Algorithm 7.3. Rectilinear Minimax Location Primal Algorithm

1. Pick Initial Radius z^0, set k = 0

2. Draw Diamonds Around P_j with radii z^k

3. Determine Intersection Z of the Diamonds

4. If Z is Empty, Increase z^k, k=k+1, Go to 2
   If Z is More than a line segment, Decrease z^k, k=k+1, Go to 2
   If Z is a Line Segment of Point, Stop

Equivalency between Chebyshev and Rectilinear Location Problems.

45 Degree Rotation and Scaling Transformation

$$
\begin{aligned}
R &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \\
S &= \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \\
Q &= RS = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\
U &= \begin{bmatrix} u \\ v \end{bmatrix} = QX = \begin{bmatrix} x+y \\ -x+y \end{bmatrix} \\
Q^{-1} &= \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} \\
X &= Q^{-1}U = \begin{bmatrix} (u-v)/2 \\ (u+v)/2 \end{bmatrix}
\end{aligned}
\tag{7.93}
$$

Chebyshev to Rectilinear Norm Equivalence

$$d_R\left(X, P_j\right) = \left|x - a_j\right| + \left|y - b_j\right| =$$

$$\max\left\{x - a_j, -x + a_j\right\} + \max\left\{y - b_j, -y + b_j\right\} =$$

$$\max\left\{x - a_j + y - b_j, x - a_j - y + b_j, -x + a_j + y - b_j, -x + a_j - y + b_j\right\} =$$

$$\max\left\{\max\left\{x - a_j + y - b_j, -x + a_j - y + b_j\right\}, \max\left\{-x + a_j + y - b_j, x - a_j - y + b_j\right\}\right\} =$$

$$\max\left\{\max\left\{x + y - a_j - b_j, -x - y + a_j + b_j\right\}, \max\left\{-x + y + a_j - b_j, x - y - a_j + b_j\right\}\right\} =$$

$$\max\left\{\left|(x+y) - (a_j + b_j)\right|, \left|(-x+y) - \left(-a_j + b_j\right)\right|\right\} =$$

$$\max\left\{\left|u - c_j\right|, \left|v - d_j\right|\right\} = d_c\left(U, Q_j\right)$$

Rectilinear minimax problem transformation

$$z = \min_X G(X,P) = \min_X \left\{ \max_j \left\{ w_j d_R(X, P_j) \right\} \right\} =$$

$$\min_U \left\{ \max_j \left\{ w_j d_C\left(U, Q_j\right) \right\} \right\} =$$

$$\min_U \left\{ \max_j \left\{ w_j \max\left\{ \left|u - c_j\right|, \left|v - d_j\right| \right\} \right\} \right\} =$$

$$\min_U \left\{ \max_j \left\{ \max\left\{ w_j \left|u - c_j\right|, w_j \left|v - d_j\right| \right\} \right\} \right\} =$$

$$\min_U \left\{ \max\left\{ \max_j \left\{ w_j \left|u - c_j\right| \right\}, \max_j \left\{ w_j \left|v - d_j\right| \right\} \right\} \right\} =$$

$$\max\left\{ \min_u \left\{ \max_j \left\{ w_j \left|u - c_j\right| \right\} \right\}, \min_v \left\{ \max_j \left\{ w_j \left|v - d_j\right| \right\} \right\} \right\} =$$

$$\max\left\{ \min_u G_u\left(u, c_j\right), \min_v G_v\left(v, d_j\right) \right\}$$

Chebyshev minisum problem transformation

$$z = \min_U F(U,Q) = \min_U \left\{ \sum_j w_j d_C\left(U, Q_j\right) \right\} =$$

$$\min_X \left\{ \sum_j w_j d_R\left(X, P_j\right) \right\} =$$

$$\min_X \left\{ \sum_j w_j \left(\left|x - a_j\right| + \left|y - b_j\right| \right) \right\} =$$

$$\min_X \left\{ \sum_j w_j \left|x - a_j\right| + \sum_j w_j \left|y - b_j\right| \right\} =$$

$$\min_x \left\{ \sum_j w_j \left|x - a_j\right| \right\} + \min_y \left\{ \sum_j w_j \left|y - b_j\right| \right\} =$$

$$\min_x F_x\left(x,a_j\right)+\min_y F_y\left(y,b_j\right)$$

Dwell Point Determination in an AS/RS rack (Simultaneous Travel)

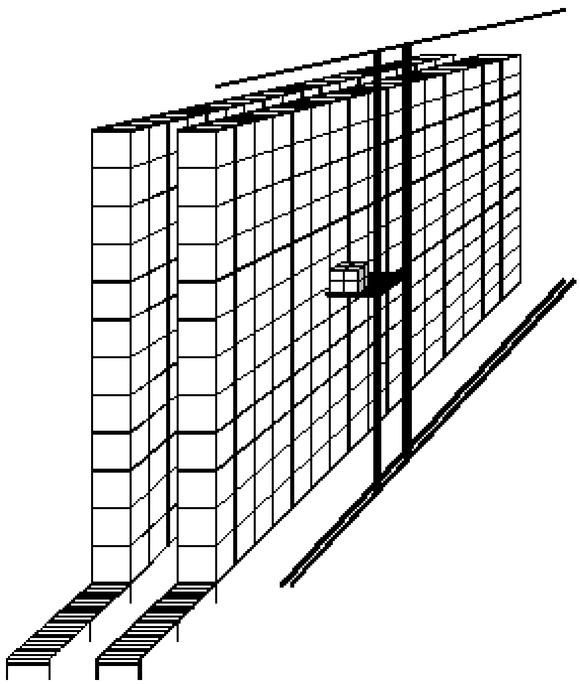


Figure 7.33. AS/RS Illustration

A dwell point policy is a set of rules determining where to position the crane when it becomes idle. Several varieties exist. The object can be to minimize the expected distance to the next service or load request location. The objective can also be to minimize the distance to the next service load request location.

The objective is to minimize the expected travel distance between the to be determined dwell point (F) and the expected events, either a storage (E) or one of four retrievals (A, B, C, or D).

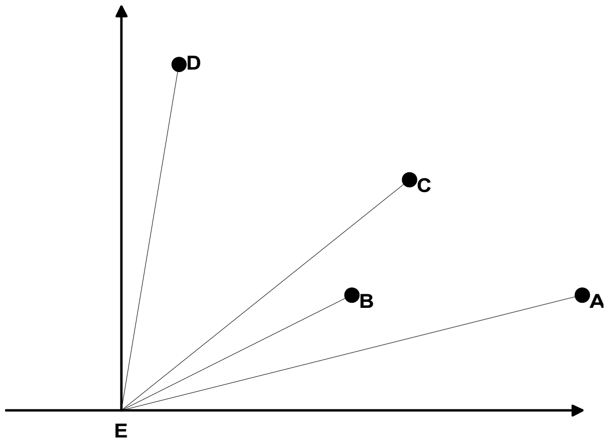


Figure 7.34. Location of Input/Output Point and Expected Retrievals in the AS/RS rack

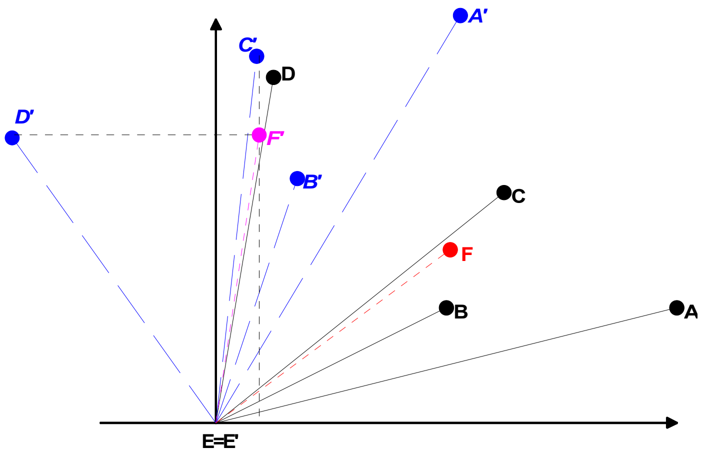


Figure 7.35. Optimal Rectilinear Minisum Location Based on the Rotated Points

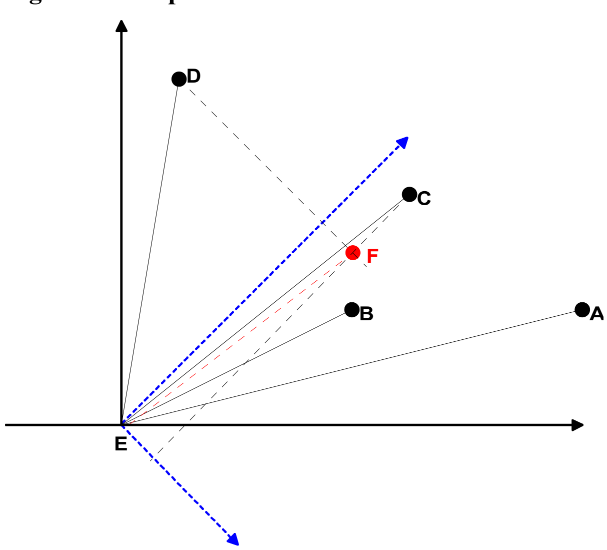


Figure 7.36. Optimal Chebyshev Minisum Location Based on the Original Points

Dwell Point Determination in an AS/RS Rack (Sequential Travel)

The objective is to minimize the maximum response time or travel distance between the to be determined dwell point (F) and the expected events, either a storage (E) or one of four retrievals (A, B, C, or D).

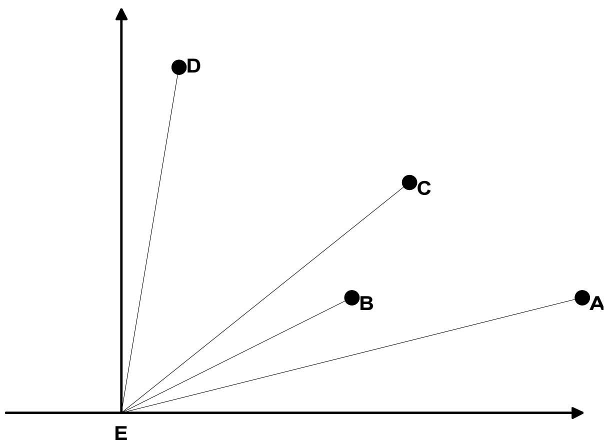


Figure 7.37. Location of Input/Output Point and Expected Retrievals in the Order Picking Rack

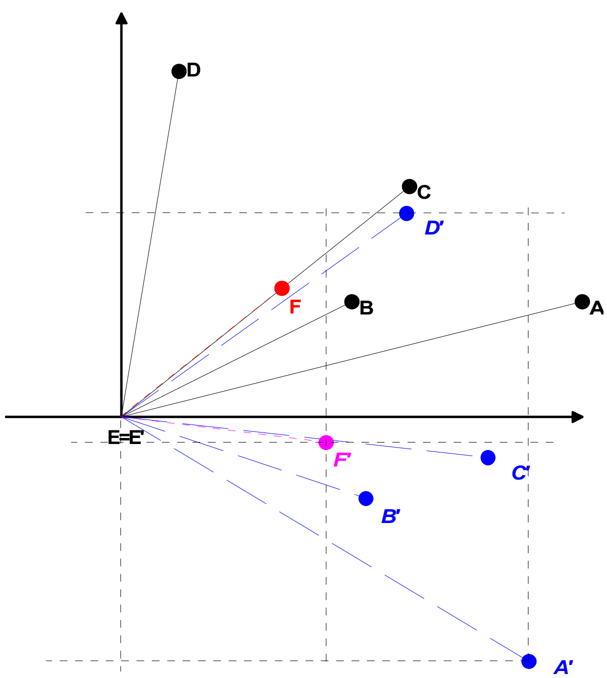


Figure 7.38. Optimal Rectilinear Minimax Location Based on the Rotated Points

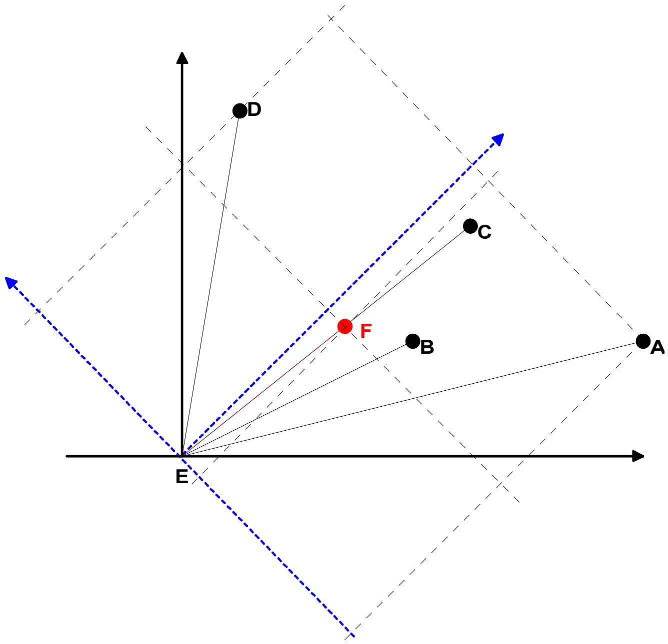


Figure 7.39. Optimal Rectilinear Minimax Location Based on the Original Points

Exercises

Euclidean Minisum Exercise 1

Consider the following Euclidean minisum multifacility facility location problem. The location of the four existing facilities is given in the following table.

Table 7.2. Existing Facilities Locations

j	a_j	b_j
1	1	2
2	2	4
3	3	3
4	4	1

The interaction between the three new facilities and the four existing facilities is given in the following table.

Table 7.3. New to Existing Facilities Interaction

	P_1	P_2	P_3	P_4
X_1	4	2	3	1
X_2	2	3	1	2
X_3	11	1	2	2

The interaction between the three new facilities is given by the following table.

Table 7.4. New to New Facilities Interaction

	X_2	X_3
X_1	1	2
X_2	-	4

Solve in the most efficient way for the optimal location of the new facilities and compute the objective function value. While solving for the optimal locations, show in a clear table your initial locations and the initial objective function. Then, execute ***one*** iteration of the iterative algorithm, if necessary. Show again in a clear table the locations of the new facilities and the objective function value. Describe clearly the assumptions and steps you have made in this algorithm.

Minimax Location Exercise

Consider the problem of finding the location of the Euclidean minimax center of a number of points with equal weight with the additional constraint that excludes certain regions for the location of the minimax center. For this particular case the gray circle represents the infeasible region for the center. The location of the points is given in the Figure 7.40.

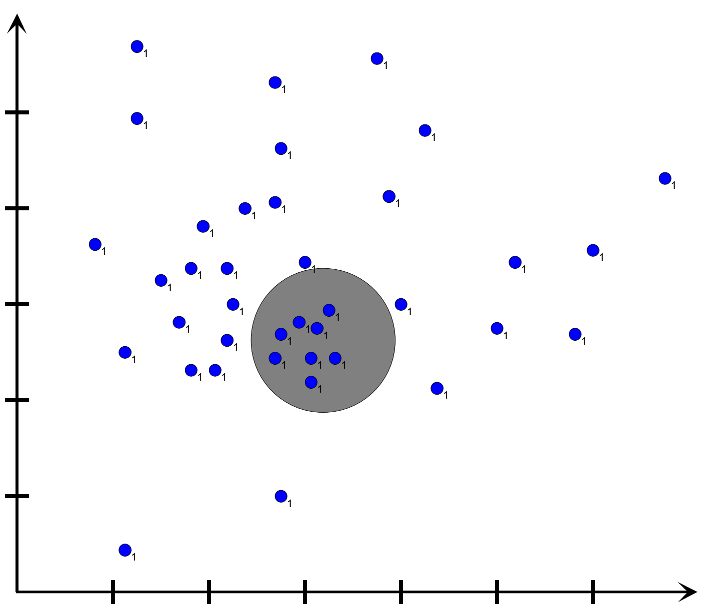


Figure 7.40. Minimax Problem with Infeasible Region

The solution is found by first finding the unconstrained optimal center with the dual Elzinga-Hearn algorithm. The location of this unconstrained center falls within the infeasible region. Observe that the intersection points of the contour lines of two points fall on the perpendicular bisector of the line connecting those two points. The intersection point of the perpendicular bisector with the infeasible region then gives the optimal constrained center location. The solution is shown in Figure 7.41. The unconstrained objective function has a value of 3.5, while the constrained objective function has a value of 3.63.

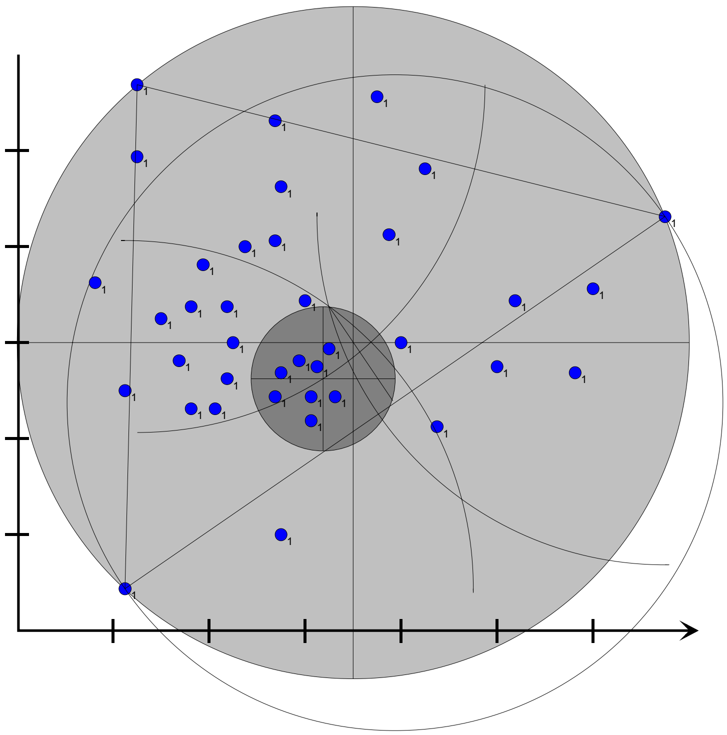


Figure 7.41. Minimax Solution with Infeasible Region

Exercise

Consider the problem of locating a single new facility in a continuous two dimensional plane. Distances are measured by the rectilinear norm. The objective is to minimize the weighted sum of distances to three existing facilities. The three existing facilities, their coordinates, and their associated weights are given in the table below.

Table 7.5. Location Data

j	a_j	b_j	w_j
1	30	20	15
2	50	30	25
3	40	20	30

Write the complete final dual programming formulation for this problem with the additional constraints $x \leq 35$ and $y \leq 25$ for the ***x component of the problem only***. Use the notation developed in class. Define clearly all the transformations and/or extra constraints that you have introduced. The dual variable associated with the extra constraint is denoted by *e* (for extra. Write out all constraints and objective function fully, i.e. without summation signs for the final dual program only.

Can the resulting dual program still be converted to a network structure? If so, draw the complete network and explain how the dual and corresponding primal solution can be found for this case.

Exercise

Consider the following rectilinear minisum multifacility facility location problem. The three existing facilities are located at $P_1(15,10)$, $P_2(5,15)$ and $P_3(10,5)$. The interactions between the four new facilities and the three existing facilities are given by the following table.

Table 7.6. New to Existing Facilities Interaction

	P_1	P_2	P_3
X_1	5	1	0
X_2	0	3	1
X_3	2	0	2
X_4	1	0	9

The interactions between the new facilities are given by the following table.

Table 7.7. New to New Facilities Interaction

	X_2	X_3	X_4
X_1	2	0	1
X_2		1	0
X_3			6

Solve in the most efficient way for the optimal location of the new facilities and compute the objective function value. Show in a clear table the optimal locations. Show also any locale networks and their corresponding cuts that you have used to derive your solution. Compute the lower and upper bound on the objective function value of the same facility location problem but with the Euclidean norm.

Exercise

Solve the following planar, multifacility rectilinear location problem. The three existing facilities are located at (3,1), (2,2), and (1,0), respectively. There are five new facilities. Show the locale network at each iteration. Give the optimal location of each new facility and the optimal objective function value. The relationships between new and old facilities are given in the matrix W, the relationships between new facilities are given in matrix V.

$$\mathbf{W} = \begin{bmatrix} 10 & 0 & 0 \\ 4 & 1 & 4 \\ 4 & 1 & 4 \\ 4 & 5 & 4 \\ 4 & 5 & 4 \end{bmatrix}$$

$$\mathbf{V} = \begin{bmatrix} - & 2 & 2 & 2 & 2 \\ 2 & - & 20 & 1 & 0 \\ 2 & 20 & - & 0 & 0 \\ 2 & 1 & 0 & - & 40 \\ 2 & 0 & 0 & 40 & - \end{bmatrix}$$

Solve the above problem for the Euclidean distance norm with the HAP procedure. The HAP procedure is available in the ISYE computer lab and documented in the ISYE software manual. Give the number of iterations, the locations of the new facilities and the objective function value.

Verify the following bounds numerically for this case. Discuss the quality of these bounds.

$$z_R(X_R^*) \geq z_E(X_E^*) \geq \sqrt{z_{RX}^2(X_{RX}^*) + z_{RY}^2(X_{RY}^*)}$$

Exercise

Consider the problem of finding the location of the rectilinear minimax center of a number of points with equal weight with the additional constraint that excludes certain regions for the location of the minimax center. For this particular case the gray rectangle represents the infeasible region for the center. The location of the points is given in the next figure. The grid lines are drawn at one-unit intervals.

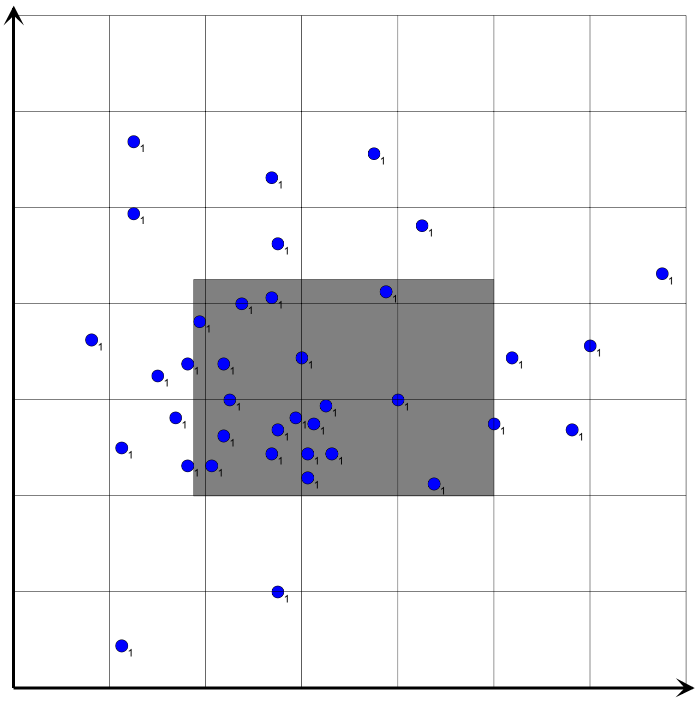


Figure 7.42. Minimax Problem with Infeasible Region

Determine first the location of the unconstrained rectilinear center, i.e. ignoring the infeasible region. Describe this location completely. What is the unconstrained minimax distance? Determine next the location of the constrained rectilinear center, i.e. observing the infeasible region. What is the constrained minimax distance.

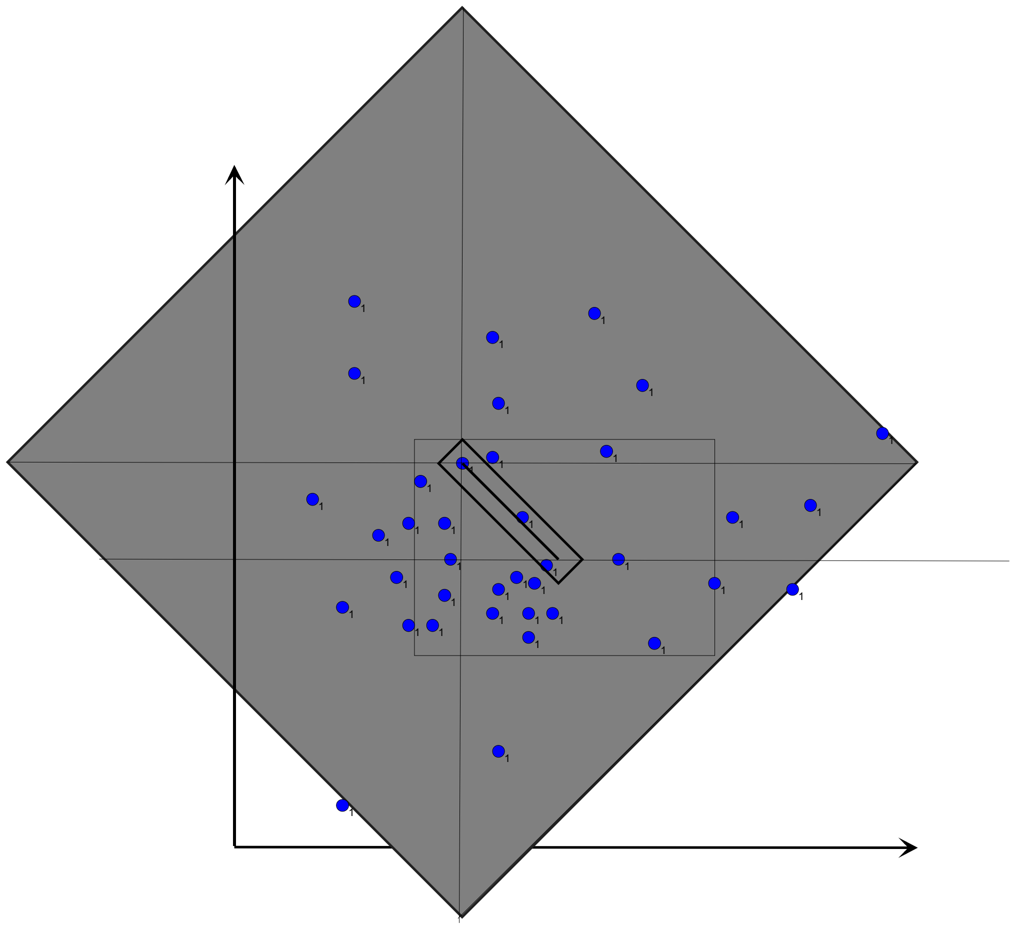


Figure 7.43. Minimax Solution with Infeasible Region

The unconstrained minimax distance is 4.75 and the alternative optimal locations of the center are given by the line segment

$$\lambda \begin{bmatrix} 2.375 \\ 4 \end{bmatrix} + (1-\lambda) \begin{bmatrix} 3.375 \\ 3 \end{bmatrix}$$

The constrained minimax distance is 5.00 and the optimal location of the constrained center is given by (2.375, 4.250).

References

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